M828

THE OPEN UNIVERSITY
MATHEMATICS TAUGHT MSc

APPLIED COMPLEX VARIABLES

Course Notes

SUP 15029 4

Course Notes for

APPLIED COMPLEX VARIABLES

Prepared by

J. G. CLUNIE

First Edition November 1985
Second Edition November 1986

Introduction

Part I of Applied Complex Variables is a largely standard introduction to the subject of complex analysis; and I shall not comment on it other than to mention that some definitions are not standard. For example, Dettman regards a subset to be closed if it is closed as a subset of the Riemann sphere (see §1.5 and Theorem 2.1.1). Note that Theorem 2.1.2 would be false if he were using closed in the usual sense - consider, for example, $f(z) = z$ on the real axis considered as a subset of the plane. However, a closed subset of the Riemann sphere is compact.

In these notes I shall act as a guide to the text and bring to your attention points which the reader might overlook; and I shall try to clarify the presentation when there seems a possibility of misunderstanding and confusion. Not all sections in Chapters 6-10 are in this course and those outside it will be clearly indicated.

To test your understanding there are Self-Assessment Questions (SAQs) and Problems. You should try all the SAQs and as many problems as you have time for. The latter are sometimes rather hard and so you should not spend too much time on any individual problem. Full solutions to SAQs and problems will be found at the end of the notes on the relevant chapter.

In order to pace you through the course we have set four Tutor-Marked Assignments (TMAs). These are compulsory in that you cannot pass the course without obtaining a reasonable average grade on them. Your three best TMAs carry 50% of the total marks for the course, the remaining 50% coming from the three-hour examination at the end of the course. Please note that we cannot accept any TMAs after the cut-off dates. These cut-off dates are as follows.

TMA 1	(Chapter 6)	17 April, 1987
TMA 2	(Chapter 7)	5 June, 1987
TMA 3	(Chapter 8, 9)	24 July, 1987
TMA 4	(Chapter 10)	25 Sept., 1987

Further information about the assessment and the telephone answering set-up will be sent to you in the various Stop Presses issued throughout the course.

It is likely that there are a number of errors in these notes, and that we have overlooked some inadequacies in Dettman's presentation. We should be grateful if you could inform us of any errors or misprints and of any suggested improvements to the Course Notes.

Chapter 6

Potential Theory

§6.1 gives a description of the origins of the mathematics of potential theory. The account is not rigorous and the presentation is rather informal in places.

| READ pages 229-232 |

p.229 The velocity v(x, y, z, t) is the same as the velocity vector $\underset{\sim}{v}$ on the next page. The physical allusions as to what v(x, y, z, t) means do not affect the mathematical status of v(x, y, z, t) which is to be thought of as a vector whose value depends upon (x, y, z) in 3-space and time t.

p.230 In

$$\frac{\Delta m}{\Delta t} = \frac{\Delta m}{\Delta v}\cdot\frac{\Delta x}{\Delta t}\cdot dS = \rho \; v_x dS$$

note that v_x is the x-component of $\underset{\sim}{v}$ and NOT a partial derivative. In passing from this equation to

$$\frac{dm}{dt} = \rho \; \underset{\sim}{v}\cdot\underset{\sim}{N} \; dS$$

it is rather confusing when one tries to think of what 'm' stands for. One can think of m as the mass of fluid collected after passing through the element Δs or dS over a period and then $\frac{\Delta m}{\Delta t}$, $\frac{dm}{dt}$ is the rate of change of this mass. Taking account of flow into V and flow out of V through dS one then obtains

$$\frac{dM}{dt} = \iint_S \rho \; \underset{\sim}{v}\cdot\underset{\sim}{N} \; dS.$$

p.231 $\underset{\sim}{\nabla}$ (nabla) the vector operator, also called grad, is given by

$$\underset{\sim}{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

in cartesian co-ordinates. Hence if $\underset{\sim}{v} = (v_x, v_y, v_z)$, then

$$\underset{\sim}{\nabla}(\rho\underset{\sim}{v}) = \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z).$$

The divergence theorem states:

$$-\iint_S \rho\underset{\sim}{v}\cdot\underset{\sim}{N} \; dS = \iiint_V \underset{\sim}{\nabla}\cdot(\rho\underset{\sim}{v}) \; dV.$$

Note that $\underset{\sim}{N}$ points inwards here, but elsewhere in J.W.D. it usually points outwards.

Assuming that $\frac{\partial \rho}{\partial t} + \nabla(\rho \underset{\sim}{v})$ is continuous the argument that leads to the continuity equation is the following. If this sum is not everywhere 0 there is a point where it is positive or a point where it is negative. By continuity there is then a neighbourhood where it is everywhere positive or everywhere negative. Taking V to be such a neighbourhood the integral over V is positive or negative and not 0. Hence the desired conclusion by reductio ad absurdum.

p.231 A domain D is simply connected if every closed curve in D can be continuously deformed within D to a point. If C is a smooth closed curve in such a domain then

$$\int_C \underset{\sim}{v}.\underset{\sim}{T}\, ds = 0$$

is analogous to Cauchy's theorem (see Theorem 3.4.1, p.92).

To obtain ϕ in this case fix $(x_0, y_0, z_0) \in D$ and for any other $(x, y, z) \in D$ let γ be a smooth path from (x_0, y_0, z_0) to (x, y, z) and set

$$\phi(x, y, z) = \int_\gamma \underset{\sim}{v}.\underset{\sim}{T}\, ds.$$

One now checks that the integral is independent of γ for (x, y, z) kept fixed and then argues as in the proof of Theorem 3.5.1, p.101 with appropriate modifications.

Self-Assessment Question

SAQ 1. Which flow is given by the velocity potential

$$\phi(x, y) = x + \frac{2x}{x^2 + y^2} ?$$

| READ pages 233-237 |

---

p.233 That heat crosses a surface element at a rate

$$k\, \underset{\sim}{\nabla} u.\underset{\sim}{N}\, dS$$

is to be taken as a hypothesis of the theory of heat transfer. $\underset{\sim}{\nabla}u.\underset{\sim}{N}$ is the rate of change of temperature normal to the surface and one is not surprised to learn that experiments show that for a given homogeneous substance the rate of heat transfer is proportional to this and that the constant of proportionality is called the thermal conductivity of the substance.

Note that heat flows from higher temperatures to lower and so $k\underset{\sim}{\nabla}u.\underset{\sim}{N}\,dS$ is the heat flow across dS in the direction $-\underset{\sim}{N}$ and $-k\underset{\sim}{\nabla}u.\underset{\sim}{N}\,dS$ is the heat flow across dS in the direction $\underset{\sim}{N}$. Hence if $\underset{\sim}{N}$ is inward pointing one sees that

$$\frac{dH}{dt} = -\iint_S k\underset{\sim}{\nabla}u.\underset{\sim}{N}\,dS$$

is correct, i.e. the negative sign is necessary.

p.234, Example 6.1.2. To say that the curved surface is perfectly insulated means that there is no heat transference across it. If u is 'smooth' across the boundary then $\frac{du}{dn} = \frac{\partial u}{\partial r} = 0$. A priori one cannot say that u is 'smooth', but one argues as if it were and then one checks any 'solution' that one obtains.

p.235, Electrostatics. Mathematically e is a real parameter. If one has charges e_1, e_2 in place of two equal charges e, then e^2 is replaced by e_1e_2.

K depends on the homogeneous substance in which the phenomena are being considered and $\frac{1}{K}$ is called the dielectric constant of the substance.

A field of force is conservative, by definition, when the work done by the force round any closed path is 0. The existence of an electrostatic potential for such a field follows in the manner outlined earlier for the existence of a velocity potential for irrotational fluid flow.

p.236, Example 6.1.3. Mathematically such questions can be considered without any special knowledge of the physical context. However, one must be able to 'interpret' the latter in mathematical terms.

p.237 $\nabla^2(\nabla^2 u) = 0$ is known as the biharmonic equation.

Self-Assessment Questions

SAQ 2. Express $\nabla^2 u = 0$ in 2 dimensions in polar co-ordinates by considering u as the real part of an analytic function and using the Cauchy-Riemann equations in polar form.

SAQ 3. Exercise 6.1, #3.

SAQ 4. Find the potential in an annulus bounded by circles of radii $a < b$ if the inner circle has potential α and the outer potential β.

| READ §6.2 |

p.237, Theorem 6.2.1 The reference to Corollary 3.7.3 in the proof is not quite in order. This corollary applies to the real and imaginary parts of an analytic function and given a harmonic function it may not be the real part of an analytic function in its domain. Consider $\log|z|$ in $\{z: 1 < |z| < 2\}$, for example. Corollary 3.7.3 is applicable to harmonic functions in simply connected domains, but does not apply to other domains such as an annulus. It is better to apply Theorem 3.7.3 and use that if u is harmonic, then u and $-u$ are subharmonic.

pp.239-240 The function

$$re^{i\theta} \mapsto \frac{R^2 - r^2}{R^2 - 2rR\cos\theta + r^2}$$

is known as the Poisson kernel for the disc $\{z: |z| < R\}$. It vanishes at every point of $\{|z| = R\}$ except at $z = R$, where it has a singularity, and has the important property of being positive for $0 < r < R$. Because of this 'good' estimates for I_1, I_2, I_3 are easily obtained.

p.239 Note that $g(\phi)$ is continuous on a compact set and so it is uniformly continuous. Hence the estimate for $|u(r, \theta) - g(\theta)|$ is uniform in θ, i.e. gives $\varepsilon > 0$ there is an $r(\varepsilon)$ $(0 < r(\varepsilon) < 1)$ such that

$$|u(r, \theta) - g(\theta)| < \varepsilon \quad (\text{all } \theta,\ r(\varepsilon) < r < 1).$$

This observation is necessary in order to establish the first sentence of Theorem 6.2.2.

p.240 The estimates for $|I_1|$, $|I_3|$ require 2M in place of M since

$$|g(\phi) - g(\theta)| \le |g(\phi)| + |g(\theta)| \le 2M.$$

p.240 For piecewise continuous see Definition 8.1.1 on p.348.

p.242 Please note that here and subsequently $\underset{\sim}{N}$ is the outward pointing normal.

p.243, Lemma 6.2.1 It is better to ignore this derivation of the Green's functions v and consider Theorem 6.3.5 and $f(z) = \dfrac{R(z - z_0)}{R^2 - \bar{z}_0 z}$

$(z_0 = x_0 + iy_0,\ z = x + iy)$.

Self-Assessment Questions

SAQ 5. What conditions on the real numbers a, b, c, d ensure that $ax^3 + bx^2y + cxy^2 + dy^3$ is harmonic?

SAQ 6. By considering $\arg(\frac{z + 1}{z - 1})$ find a function g harmonic in the unit disc such that

$$g(e^{i\theta}) = \begin{matrix} 1 (0 < \theta < \pi) \\ -1 (-\pi < \theta < 0). \end{matrix}$$

| READ §6.3 |

p.245 The essential feature of a Green's function $G(x, y, x_0, y_0)$ is the singular behaviour specified by $-\frac{1}{2\pi}\log|z - z_0|$. The behaviour on C is specified so that an application of Green's theorem (Exercises 3.1, #4) leads to the desired result.

pp.251-252 The argument is presented in an unfortunate way. Notation is specified subsequent to its use - an undesirable feature in any presentation. Perhaps taking one of the two domains to be a disc rather than keeping both domains general hinders the understanding.

Exercises 6.3, #1

This seems a rather difficult problem. To show uniqueness we must show that if $g(x) \equiv 0$, then $u(z) \equiv 0$ in the upper half-plane. Assume then that $g(x) \equiv 0$.

We can extend u to be a harmonic function in the plane by setting $u(z) = - u(\bar{z})$ for z in the lower half plane. From Green's theorem (Ex 4, p.80) we obtain

$$2\int_0^{\pi} \left(u \frac{\partial u}{\partial r}\right)_{r=R} R\, d\theta = \int_0^{2\pi} \left(u \frac{\partial u}{\partial r}\right)_{r=R} R\, d\theta = \iint_{|z|\le R} (u\nabla^2 u + |\nabla u|^2)\, dxdy.$$

Note that $\nabla u.\underset{\sim}{N}$ is the derivative of u in the direction of the unit outward pointing normal $\underset{\sim}{N}$ and so in this case $\nabla u.\underset{\sim}{N} = \frac{\partial u}{\partial r}$.

Since u is harmonic $\nabla^2 u = 0$ and hence

$$\int_0^{\pi} \left(u \frac{\partial u}{\partial r}\right)_{r=R} R\, d\theta = \frac{1}{2} \iint_{|z|\le R} |\nabla u|^2\, dxdy.$$

Fixing $R_0 (0 < R_0 < 1)$ we deduce that for $R \ge R_0$,

$$\iint_{|z|\le R_0} |\nabla u|^2\, dxdy \le \iint_{|z|\le R} |\nabla u|^2\, dxdy = 2\int_0^{\pi} \left(u \frac{\partial u}{\partial r}\right)_{r=R} R\, d\theta \to 0 \ (R \to \infty).$$

Therefore $\iint_{|z|\le R_0} |\nabla u|^2\, dxdy = 0$ and so $\nabla u \equiv 0$, and consequently u is constant. u is 0 on the x-axis and so $u(z) \equiv 0$.

Self-Assessment Question

SAQ 7. By considering the definition of a Green's function on p.245 and Theorem 6.3.4 prove the result given in Exercises 6.3, #2.

Problems for §6.1 - §6.3

P1. Consider a 2-dimensional fluid flow outside a simple curve C in the (x, y) - plane. If the velocity $\underset{\sim}{v} = (v_x, v_y)$ is such that $v_u - iv_y$ is analytic outside C, i.e. $v_x - iv_y = f(z)$ with $z = x + iy$ and $f(z)$ analytic, show that the circulation is the same round any simple curve surrounding C.

P2. Consider the hypotheses of the preceding problem with C the unit circle $\{|z| = 1\}$. Prove that if the circulation is non-zero then f(z) is of the form $\sum_{-\infty}^{\infty} a_k z^k$ with $\text{Im } a_{-1} \neq 0$.

P3. Find a steady state temperature for a thin metal infinite strip $\{(x, y): -\infty < x < \infty, 0 \leq y \leq 1\}$ when the upper edge is perfectly insulated and $u(-1, 0) = 3$, $u(1, 0) = 2$. How many solutions are there for u(x, y)?

P4. Prove the assertion on p.235 that

$$\int_C \frac{Ke^2}{r^3} (xdx + ydy + zdz)$$

is independent of the path assuming that the path is smooth.

P5. If u(z) is harmonic and positive in the unit disc and continuous in the closed unit disc show that

$$u(0) \frac{1 - r}{1 + r} \leq u(re^{i\theta}) \leq u(0) \frac{1 + r}{1 - r} \quad \text{(Harnack's inequalities).}$$

[Hint: first find upper and lower bounds for the Poisson kernel of the unit disc.]

P6. Let D be a domain in the plane bounded by a simple smooth curve C. Let g be a real-valued smooth function defined on C. F is the family of real-valued functions with smooth derivatives on D and C and boundary values g. If

$$\inf\{\iint_D (u_x^2 + u_y^2)\, dA: u \in F\}$$

is attained by $U \in f$ show that U is the solution of the Dirichlet problem for D with boundary values g.

(Dirichlet's Principle)

| READ §6.4 |

Definition 6.4.1 f(z) would nowadays be called univalent, or perhaps schlicht, rather than simple.

Theorem 6.4.4 The convergence referred to is defined as follows: given any compact $K \subseteq D$, $f_n(z) \to f(z)$ uniformly on K, i.e. given $\varepsilon > 0$, there is an n_0 so that

$$|f_n(z) - f(z)| < \varepsilon \quad (\text{all } z \in K,\ n > n_0).$$

Note that n_0 depends not only on ε, but also on K.

Theorem 6.4.7 This is one of the most important theorems of complex analysis. You should try and get some idea of how the proof goes without being too concerned with details. Roughly speaking the steps of the proof are as follows.

(i) Let G be the family of functions univalent (i.e. simple) and having moduli bounded by 1 in D.

(ii) Prove that G is non-empty, i.e. it contains at least one function. (This is essential for any subsequent arguments to have substance.)

(iii) Consider the extremal problem

$$M = \sup\{|g'(z_0)| : g \in G\}$$

with $z_0 \in D$ fixed.

(iv) Show that there is an extremal function $f \in G$ with $|f'(z_0)| = M$.

(v) Check that f has the desired properties.

One of the common reactions to this proof is: Why consider (iii)? If one considers a trial and error approach in proving the Riemann Mapping Theorem, then one is looking for promising methods that may turn out to be effective. If k is analytic then $|k'(z_0)|$ is the local magnification factor of the mapping $z \mapsto k(z)$ at $z = z_0$. The larger $|k'(z_0)|$ is, then the larger the magnification at $z = z_0$. In the Riemann Mapping Theorem we are trying to find a mapping f whose image fills out all of the unit disc, i.e. in some global sense we are trying to make the mapping as large as possible. However, by fixing a point $z_0 \in D$ and making the magnification at z_0 as large as possible subject to (i) one might hope to optimise in a global sense, as turns out to be the case.

Remember that when studying mathematics what we read is often the successful outcome of a trial and error approach in the past. The development has often been quite chaotic and the orderliness of the way it is presented in text-books should be recognised for the sythesis it so often is.

Self-Assessment Questions

SAQ 8. Exercises 6.4, #1.

SAQ 9. Exercises 6.4, #2.

SAQ 10. Exercises 6.4, #3.

| READ § 6.5 |

In dealing with Schwarz-Christoffel transformations note that we know the mappings exist by the Riemann Mapping Theorem and Theorem 6.4.8. Consequently the existence of the mapping in any particular instance is not in question and the problem is to characterise it. This is easier than proving its existence ab initio and then determining its properties. It is possible to do this without first establishing the Riemann Mapping Theorem and Theorem 6.4.8.

Schwarz - Christoffel transformations are deceptively simple. Given a polygon P let $z = f(w)$ be the Schwarz-Christoffel transformation of the upper half w-plane onto P. To determine f requires finding the location of the points on the real w-axis which map onto the vertices of P. This is however an extremely difficult problem even for quite simple polygons P.

When one deals with Schwarz-Christoffel transformations one can either consider the mapping from the upper half plane to the polygon or the other way round. It is nearly always better to consider the former mapping and not the latter.

The examples and discussion on pp.260-265 show how one deals with Schwarz-Christoffel transformations, i.e. one considers what must be the form of $\frac{dz}{dw}$ and then finds $z = f(w)$ by integration. The upper half plane has a boundary point at $w = \infty$ and this may correspond to a point on a side of the polygon or to a vertex of the polygon. If one finds the form of $\frac{dz}{dw}$ by considering those points on the real w-axis corresponding to vertices, then the correct behaviour at $w = \infty$ will have been determined thereby without $w = \infty$ having to be specially considered.

Problems for §6.4–§6.5

P7. Find a univalent mapping of the half-disc $\{z: |z| < 1, \text{Im } z > 0\}$ onto the unit disc.

P8. Let $f(z)$ be univalent in $\{|z| < 1\}$ with $f(0) = 0$. Show that one can find $g(z)$ univalent in the unit disc and satisfying

$$(g(z))^2 = f(z^2) \quad (|z| < 1).$$

P9. Let $f(z)$ be analytic and map the annulus $\{1 < |z| < R_1\}$ univalently onto $\{1 < |w| < R_2\}$ so that the inner and outer boundaries correspond. Show that $f(z) = Az$ for some constant A with $|A| = 1$ and so $R_1 = R_2$. What happens when $\{|z| = R_1\}$ maps to $\{|w| = 1\}$ and $\{|z| = 1\}$ to $\{|w| = R_2\}$?

P10. Exercises 6.5, #1.

P11. Exercises 6.5, #2.

| READ §6.6–§6.7 |

p.267 Physically the flows dealt with are considered as taking place in $\mathbb{R}^3$ and when we restrict attention to $\mathbb{R}^2$ we regard the flow as being independent of vertical height. Later in §6.7 when one talks about a 'line source' this is to be regarded as a source uniform along a vertical line.

The formula given for fluid flowing in Example 6.6.1 is in fact the amount of fluid per unit height. Note that a similar interpretation must be given throughout this section for the results to be thought of as physically significant. Flows depending on height are not dealt with.

Self-Assessment Questions

SAQ 11. Exercises 6.6, #3.

SAQ 12. Exercises 6.6, #4.

SAQ 13. Exercises 6.7, #2.

§6.8 is not part of M828

Solutions to Self-Assessment Questions

1. A uniform stream flowing past a circular cylinder of radius $\surd 2$.

2. Let $u(z) = \text{Re } f(z)$ and $u(z) + iv(z) = f(z)$. In terms of polars the Cauchy-Riemann equations are

$$r\,\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta},\quad r\,\frac{\partial v}{\partial r} = -\,\frac{\partial u}{\partial \theta}.$$

Therefore

$$\frac{\partial}{\partial r}(r\,\frac{\partial u}{\partial r}) = \frac{\partial}{\partial r}(\frac{\partial v}{\partial \theta}) = \frac{\partial}{\partial \theta}(\frac{\partial v}{\partial r}) = -\,\frac{\partial}{\partial \theta}(\frac{1}{r}\,\frac{\partial u}{\partial \theta})$$

and so Laplace's equation is

$$\frac{\partial}{\partial r}(r\,\frac{\partial u}{\partial r}) + \frac{\partial}{\partial \theta}(\frac{1}{r}\,\frac{\partial u}{\partial \theta}) = 0.$$

3. $\phi(re^{i\theta}) = 10 \log \frac{r}{a}/\log \frac{b}{a}$.

4. $\phi(re^{i\theta}) = \dfrac{(\beta - \alpha) \log r + \alpha \log b - \beta \log a}{\log b/a}$.

5. From $\nabla^2\phi = 0$, $3a + c = 0$ and $b + 3d = 0$ so that

$$\begin{aligned}\phi(x, y) &= ax^3 - 3dx^2y - 3axy^2 + dy^3\\ &= a(x^3 - 3xy^2) + d(-3x^2y + y^3)\\ &= a \text{ Re } z^3 + d \text{ Im } z^3 \quad (z = x + iy).\end{aligned}$$

6. We can define $\arg(\frac{z+1}{z-1})$ continuously in $\{|z| < 1\}$ so that

$$\arg(\frac{e^{i\theta}+1}{e^{i\theta}-1}) = \begin{cases} -\frac{\pi}{2} & (0 < \theta < \pi \\ \frac{\pi}{2} & (\pi < \theta < 0). \end{cases}$$

Therefore

$$g(z) = -\,\frac{2}{\pi}\arg(\frac{z+1}{z-1})$$

since $\arg(\frac{z+1}{z-1}) = \text{Im}[\log(\frac{z+1}{z-1})]$.

7. If $z = x + iy$, $z_0 = x + iy_0$, then

$$\log \surd(x - x_0)^2 + (y - y_0)^2 = \log|z - z_0| = \text{Re } \log(z - z_0)$$

$$\log \surd(x - x_0)^2 + (y + y_0)^2 = \log|z - \bar{z}_0| = \text{Re } \log(z - \bar{z}_0)$$

and so $G(x, y, x_0, y_0)$ is harmonic in (x, y) for fixed (x_0, y_0) except at (x_0, y_0), $(x_0, -y_0)$. Also

$$G(x, 0, x_0, y_0) = 0.$$

Assuming that Theorem 6.3.4 is applicable in this case we find that

$$u(x_0, y_0) = -\int_{-\infty}^{\infty} g(x).\ \nabla G.\underset{\sim}{N}\ dx$$

$$= -\int_{-\infty}^{\infty} g(x)\ [-\frac{\partial}{\partial y}(-\frac{1}{4\pi}\log((x-x_0)^2 + (y-y_0)^2)$$

$$+ \frac{1}{4\pi}\log((x-x_0)^2 + (y+y_0)^2))]_{y=0}\ dx$$

$$= -\frac{1}{4\pi}\int_{-\infty}^{\infty} g(x)\ [-\frac{2\,y_0}{(x-x_0)^2 + y_0^2} - \frac{2\,y_0}{(x-x_0)^2 + y_0^2}]\ dx$$

$$= \frac{y_0}{\pi}\int_{-\infty}^{\infty} \frac{g(x)}{(x-x_0)^2 + y_0^2}\ dx.$$

8. For z_1, z_2 in $\{|z| < 1\}$ and $z_1 \neq z_2$,

$$f(z_1) - f(z_2) = (z_1 - z_2)\ \{1 + a_2(z_1 + z_2) + a_3(z_1^2 + z_1z_2 + z_2^2) + \ldots]$$

and so

$$|f(z_1)-f(z_2)| \geq |z_1-z_2|[1-|a_2|(|z_1|+|z_2|)-|a_3|(|z_1|^2+|z_1z_2|+|z_2|^2)-\ldots]$$

$$\geq |z_1 - z_2|\ [1 - 2|a_2| - 3|a_3|\ \ldots] > 0.$$

Hence if $z_1 \neq z_2$. then $f(z_1) \neq f(z_2)$ and so $f(z)$ is univalent in the unit disc.

9. (a) If $z \in C$ then the element of length at z along C is $|dz|$. This element of length is transformed by f into an element of length along C^* given by $|f'(z)||dz|$ at $f(z) \in C^*$. Hence

$$\text{length of } C^* = \int_C |f'(z)||dz|.$$

(b) If $z = x + iy \in D$, then the element of area at z is $dxdy$. This is transformed by f into an element of area of D^* given by $|f'(z)|^2$ $dxdy$ at $f(z) \in D^*$. Hence

$$\text{area of } D^* = \iint_D |f'(z)|^2\ dxdy.$$

(c) If $f(z) = \sum_0^{\infty} a_kz^k\ (|z| < R)$, then $f'(z) = \sum_1^{\infty} ka_kz^{k-1}$. Consequently

$$|f'(re^{i\theta})|^2 = \sum_1^{\infty} ka_kr^{k-1}e^{i(k-1)\theta}\ .\ \sum_1^{\infty} k\bar{a}_kr^{k-1}.e^{-i(k-1)\theta}\ .$$

Hence

$$\text{area of } D^* = \iint_{|z|<R} |f'(re^{i\theta})|^2 \, r dr d\theta$$

$$= \int_0^R r dr . \int_0^{2\pi} \left(\sum_1^\infty k a_k r^{k-1} e^{i(k-1)\theta} \sum_1^\infty k \bar{a}_k r^{k-1} e^{-i(k-1)\theta}\right) d\theta$$

$$= 2\pi \int_0^R \sum_1^\infty k^2 |a_k|^2 \, r^{2k-2} . \, r dr,$$

multiplying up and integrating term by term and using

$$\int_0^{2\pi} e^{ik\theta} d\theta = \begin{cases} 1 & (k = 0) \\ 0 & (k \neq 0). \end{cases}$$

Therefore

$$\text{area of } D^* = \pi \sum_1^\infty k|a_k|^2 \, R^{2k}$$

$$\geq \pi \, |a_1|^2 \, R^2 = \pi \, |f'(0)|^2 \, R^2.$$

10.. In place of $3z + z^2$ one can consider $z + \frac{1}{3} z^2$. In this case we see that in the notation of SAQ 1. $\sum_{k=2}^{\infty} k|a_k| = \frac{2}{3} < 1$ and so univalency follows from the earlier result.

11. To say that a flow has a source of strength A at $z = z_0$ means that in some neighbourhood of z_0 the flow is given by a complex velocity potential of the form

$$w(z) = A \log (z - z_0) + g(z) \quad (A > 0),$$

where $g(z)$ is analytic in $\{0 < |z - z_0| < \rho\}$ for some $\rho > 0$. If C is a simple closed curve in this punctured disc that surrounds $z = z_0$ in the positive direction then the rate of the flow across C is, apart from fluid density,

$$\frac{1}{i} \int_C \left(\frac{A}{z - z_0} + g'(z)\right) dz = 2\pi A.$$

Here we have put $\int_C \underset{\sim}{v}.\underset{\sim}{N} \, ds$ into complex form.

Let $f(z) = \zeta$ be a 1 - 1 conformal transformation of a neighbourhood of z_0, i.e. in some neighbourhood $f(z)$ is univalent and so $f'(z) \neq 0$ in this neighbourhood. If $z = F(\varsigma)$ and $\varsigma_0 = f(z_0)$ then $F(\varsigma)$ is 1 - 1 conformal in a neighbourhood of ς_0 and so $F'(\varsigma) \neq 0$ in this neighbourhood. The transformed flow at ς_0 is

$$w(\varsigma) = w(F(\varsigma)) = A \log(F(\varsigma) - F(\varsigma_0)) + g(F(\varsigma))$$

Since $F(\varsigma)$ is univalent in the neighbourhood of ς_0 and $F(\varsigma_0) = z_0$ it follows that in some punctured disc $\{0 < |\varsigma - \varsigma_0| < \sigma\}$, $F(\varsigma) \neq z_0$ and so in this punctured disc $g(F(\varsigma))$ is analytic. If $F(\varsigma) = \sum_0^\infty a_k(\varsigma - \varsigma_0)^k$ $(|\varsigma - \varsigma_0| < \sigma)$ then

$$F(\varsigma) - F(\varsigma_0) = F(\varsigma) - a_0$$

$$= a_1(\varsigma - \varsigma_0) [1 + \sum_2^\infty \frac{a_k}{a_1} (\varsigma - \varsigma_0)^{k-1}]$$

since $a_1 = F'(\varsigma_0) \neq 0$. Therefore

$$\log(F(\varsigma) - F(\varsigma_0)) = \log(\varsigma - \varsigma_0) + \log a_1 + \log[1 + \sum_2^\infty \frac{a_k}{a_1} (\varsigma - \varsigma_0)^{k-1}]$$

$$= \log(\varsigma - \varsigma_0) + \text{function analytic in a neighbourhood of } \varsigma_0.$$

Hence

$$W(\varsigma) = A \log(\varsigma - \varsigma_0) + \text{function analytic in a punctured disc about } \varsigma_0.$$

This shows that a source is conformally invariant.

12. We shall only deal with the essential component of a doublet in a flow, i.e.

$$w(z) = -\frac{e^{i\alpha}M}{z - z_0}.$$

If $\varsigma = f(z)$, $z = F(\varsigma)$ is the same as above then

$$W(\varsigma) = w(F(\varsigma)) = -\frac{e^{i\alpha}M}{F(\varsigma) - F(\varsigma_0)}$$

Assuming $F(\varsigma)$ is given as in the earlier solution we see that

$$W(\varsigma) = -\frac{e^{i\alpha}M}{a_1(\varsigma - \varsigma_0)} + \text{function analytic at } \varsigma_0.$$

Since $a_1 = F'(\zeta_0) = \dfrac{1}{f'(z_0)}$ we obtain the result of the question. Setting $f'(z_0) = |f'(z_0)|e^{i\beta}$ we see that the direction of the doublet is changed from α to

$$\alpha + \beta = \alpha + \arg f'(z_0).$$

13. Considering Theorem 6.2.2 and the result for the integral of the Poisson kernel on p.289 we find that if we take $z_0 = 0$, which we can do without loss of generality,

$$\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})r\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - \phi) + r^2}\,r\,d\theta\right]u(Re^{i\phi})\,d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} r\,u(Re^{i\phi})\,d\phi.$$

Hence

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{R} u(re^{i\theta})r\,dr\,d\theta = \int_0^{R} r\,dr . \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\phi})\,d\phi$$

$$= \frac{R^2}{2} . u(0) \quad \text{(from Theorem 6.2.2),}$$

and so

$$u(0) = \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^{R} u(re^{i\theta})r\,dr\,d\theta$$

$$= \frac{1}{\pi R^2}\iint_D u\,dx\,dy \quad (D = \{|z| < R\}).$$

Solutions to Problems

1. Let C_1, C_2 be two simple closed curves surrounding C in the positive direction with one, C_1 say, lying inside the other. Now we can take $\underset{\sim}{v} = (v_x, v_y)$ and (dx, dy) as $\underset{\sim}{T}.ds$ on a curve so that

$$\underset{\sim}{v}.\underset{\sim}{T}ds = v_x dx + v_y dy$$

$$= \mathrm{Re}[(v_x - iv_y)(dx + idy)]$$

$$= \mathrm{Re}[f(z)dz],$$

where $f(z)$ is the analytic function $v_x - iv_y$ with $z = x + iy$. By Theorem 3.4.5, p.97 it follows that

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

and so

$$\int_{C_1} \underset{\sim}{v}.\underset{\sim}{T}ds = \int_{C_2} \underset{\sim}{v}.\underset{\sim}{T}ds.$$

If C_1, C_2 intersect one chooses a curve Γ that is disjoint from both C_1 and C_2 and compares circulation round C_1 with that round Γ; and that round C_2 with that round Γ and so one gets the general result.

2. Since f(z) is analytic in $\{|z| > 1\}$ it has a representation for such z of the form

$$f(z) = \sum_{-\infty}^{\infty} a_k z^k \quad \text{(see § 4.6).}$$

Hence the circulation round $\{|z| = 1\}$ is

$$\begin{aligned} \mathrm{Re}[\int_{|z|=2} f(z)dz] &= \mathrm{Re}[\int_0^{2\pi} (\sum_{-\infty}^{\infty} a_k 2^k e^{ik\theta}) i\ 2e^{i\theta} d\theta\,] \\ &= \mathrm{Re}[i \sum_{-\infty}^{\infty} a_k 2^{k+1} \int_0^{2\pi} e^{i(k+1)\theta} d\theta\,] \\ &= \mathrm{Re}[2\pi i a_{-1}] \end{aligned}$$

since for any non-zero integer n,

$$\int_0^{2\pi} e^{in\theta} d\theta = 0.$$

Consequently the circulation is non-zero if and only if Im $a_{-1} \neq 0$.

3. We are looking for a harmonic function u(x, y) in the strip such that

$$\frac{\partial}{\partial y} u(x, y)\Big|_{y=1} = 0 \qquad (-\infty < x < \infty)$$

$$u(-1, 0) = 3, \quad u(1, 0) = 2.$$

The upper edge condition suggests looking for u(x, y) which is a function of x alone. IF u(x, y) = g(x), then $\nabla u = 0$ in the strip becomes

$g''(x) = 0$ and so $g(x) = Ax + B$, for constants A and B. Then

$u(-1, 0) = -A + B = 3,$

$u(1, 0) = A + B = 2$

and so

$$B = \frac{5}{2}, \ A = -\frac{1}{2}.$$

One now sees that

$$u(x, y) = -\frac{x}{2} + \frac{5}{2}$$

is a solution to the problem.

Another possibility is that one looks for u(x, y) of the form g(x) h(y) with h'(1) = 0. Consider c real and

$$\mathrm{Re}[e^{cz-ic}] = \mathrm{Re}[e^{cx+ic(y-1)}]$$
$$= e^{cx} \cos c(y - 1).$$

and note that

$$\frac{d}{dy} \cos c(y - 1) = -c.\sin c(y - 1)$$
$$= 0 \ (y = 1).$$

If we choose $c = \frac{\pi}{2}$, then when y = 0,

$$e^{\frac{\pi x}{2}} \cos c(y - 1) = e^{\frac{\pi x}{2}} \cos \frac{\pi}{2} = 0$$

and so

$$u(x, y) = -\frac{x}{2} + \frac{5}{2} + Ae^{\frac{\pi x}{2}} \cos(\tfrac{\pi}{2}(y - 1))$$

solves the problem for any constant A and therefore there are infinitely many solutions.

4. A path C is given parametrically as (x(u), y(u), z(u)) for a ≤ u ≤ b and to be smooth means that x(u), y(u), z(u) are nicely differentiable. Then on the path

$$xdx + ydy + zdz = (x(u)x'(u) + y(u)y'(u) + z(u)z'(u))du$$

and so

$$\int_C \frac{Ke^2}{r^3} (xyx + ydy + zdz) = Ke^2 \int_a^b \frac{x(u)x'(u) + y(u)y'(u) + z(u)z'(u)}{(x^2(u) + y^2(u) + z^2(u))^{3/2}} du$$
$$= Ke^2 . - \frac{1}{(x^2(u) + y^2(u) + z^2(u))^{1/2}}\Big|_a^b .$$

Given that one end point of the path (x(a), y(a), z(a)) is fixed then we see that

$$\int_C \frac{Ke^2}{r^3} (xdx + ydy + zdz)$$

depends only on (x(b), y(b), z(b)), the other end of C, and so the result follows.

5. Note first that

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \phi) + r^2} u(e^{i\phi})d\phi$$

and so

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\phi})d\phi.$$

Since the Poisson kernel is positive and $U(e^{i\phi})$ is positive we see that

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\min_{0\le t\le 2\pi}\frac{1 - r^2}{1 - 2r\cos t + r^2}\right)u(e^{i\phi})d\phi \le u(re^{i\theta})$$

$$\le \frac{1}{2\pi}\int^{2\pi}\left(\max_{0\le t\le 2\pi}\frac{1 - r^2}{1 - 2r\cos t + r^2}\right) u(e^{i\phi})d\phi$$

and so

$$\frac{1 - r^2}{(1 + r)^2}\cdot\frac{1}{2\pi}\int_0^{2\pi} u(e^{i\phi})d\phi \le u(re^{i\theta}) \le \frac{1 - r^2}{(1 - r)^2}\cdot\frac{1}{2\pi}\int_0^{2\pi} u(e^{i\phi})d\phi$$

i.e. $\frac{1 - r}{1 + r}\cdot u(0) \le u(re^{i\theta}) \le \frac{1 + r}{1 - r}\cdot u(0).$

6. Let U be a function in F for which the infimum is attained. Suppose that v is smooth in D and on C and 0 every where on C. Then for all $\eta \in \mathbb{R}$, $U + \eta\, v \in F$ and so

$$\iint_D (U_x^2 + U_y^2)dxdy \le \iint_D [(U_x + \eta v_x)^2 + (U_y + \eta v_y)^2]dxdy.$$

Therefore

$$2\eta\iint_D (U_x v_x + U_y v_y)dxdy + \eta^2\iint_D (v_x^2 v_y^2)dxdy \ge 0.$$

If the first integral above is non-zero then by choosing η small and of appropriate sign we can make the left hand side above negative, which is a contradiction. Hence for all such v,

$$\iint_D \nabla v.\nabla U\, dxdy = \iint_D (U_x v_x + U_y v_y)dxdy = 0.$$

Consider now the first form of Green's identity in # 4, p.80. With U, v as above we see that

$$\iint_D (v\nabla^2 U + \nabla v.\nabla U)dxdy = \int_C v\nabla U.\underset{\sim}{N}ds.$$

Since $v = 0$ on C and $\iint_D \nabla v.\nabla U\, dxdy = 0$ we see that

$$\iint_D v\nabla^2 U\, dxdy = 0.$$

Suppose now that $\nabla^2 U$ is continuous in D and at some point $(x_0, y_0) \in D$ is non-zero. Then for some $\varepsilon > 0$, $d > 0$, $\{|z - z_0| \le 2d\} \in D$, where $z_0 = x_0 + iy_0$, and $\nabla^2 U > \varepsilon$ in this closed disc or $\nabla^2 U < -\varepsilon$ in the disc. Define v by

$$v(z) = \begin{cases} 1 & (|z - z_0| \le d) \\ (r - d)^3 (2d - r)^3 & (d \le |z - z_0| \le 2d, \ |z - z_0| = r) \\ 0 & (|z - z_0| \ge 2d). \end{cases}$$

This particular v satisfies the requirements of smoothness and being 0 on C and so

$$0 = \left|\iint_D v \nabla^2 U \, dxdy\right| = \left|\iint_{|z-z_0|\le 2d} v \nabla^2 U \, dxdy\right|$$

$$\ge \varepsilon \iint_{|z-z_0|\le d} dxdy$$

$$= \varepsilon . \pi d^2 > 0.$$

This is a contradiction and so $\nabla^2 U = 0$ in D, i.e u is harmonic in D and therefore solves the [Dirichlet] problem for D with boundary values g.

7. Consider §2.4, p.43 on Mobius (linear fractional) mappings. In this problem the boundary of the domain consists of two circular arcs if we think of a straight line as a circle through $z = \infty$. The picture is clearer if we consider the Riemann sphere rather than the complex plane. If we employ the mapping $w = \frac{1 + z}{1 - z}$, then in the w-plane the image will be bounded by arcs of circles through $z = \infty$, i.e. straight lines and intersecting at $z = 0$. Furthermore the angle between the arcs of the boundary at $z = -1$ is $\frac{\pi}{2}$ and so the image arcs or segments will also be at right-angles.

Clearly the segment between -1, 1 of the real z-axis maps onto the positive real w-axis. By conformality, or by direct verification, the semi-circular maps onto the positive imaginary w-axis and so the half-disc in the z plane maps onto the first quadrant of the w-plane.

The mapping $W = w^2$ maps the first quadrant in the w-plane onto the upper half W-plane and $\zeta = \frac{W - i}{W + i}$ maps the upper half W-plane onto the unit ζ-disc about $\varsigma = 0$. Combining these mappings one finds that a mapping of the kind asked for is

$$\varsigma = \frac{W - i}{W + i} = \frac{w^2 - i}{w^2 + 1} = \frac{(\frac{1 + z}{1 - z})^2 - i}{(\frac{1 + z}{1 - z})^2 + i} = \frac{(1 + z)^2 - i(1 - z)^2}{(1 + z)^2 + i(1 - z)^2}.$$

8. If $f(0) = 0$, then $f(z) = z F(z)$, where $F(z)$ is analytic in $\{|z| < 1\}$. Since $f(z)$ is univalent in the unit disc $F(z)$ has no zero in $\{|z| < 1\}$. $F(z)$ cannot be 0 for $z \neq 0$ and if $F(0) = 0$, then $f'(0) = 0$ and then $f(z)$ takes values at least twice in a neighbourhood of $z = 0$. (See Theorem 6.4.1). Hence $F(z)$ has an analytic logarithm in $\{|z| < 1\}$, i.e. there is a function $h(z)$ analytic in the unit disc such that

$$F(z) = e^{h(z)} \quad (|z| < 1).$$

Consequently, $F(z)$ has an analytic square root

$$G(z) = \exp\left[\frac{h(z)}{2}\right]$$

since

$$(G(z))^2 = e^{h(z)} = F(z).$$

Consider now $g(z) = z\,G(z^2)$. $g(z)$ is analytic in $\{|z| < 1\}$ and

$$(g(z))^2 = z^2(G(z^2))^2 = z^2F(z^2) = f(z^2).$$

Also $G(z^2)$ is even, i.e. $G(z^2) = G((-z)^2)$, and $g(z)$ is odd, i.e. $g(-z) = -g(z)$. Suppose $g(z)$ were not univalent in the unit disc. Then for some z_1, z_2 in $\{|z| < 1\}$, $z_1 \neq z_2$, $g(z_1) = g(z_2)$ and so $f(z_1^2) = f(z_2^2)$. As $f(z)$ is univalent we must have $z_1^2 = z_2^2$. Hence $z_1 = z_2$ or $z_1 = -z_2$. Assuming as we are that $z_1 \neq z_2$, we see that $z_1 \neq 0$ and $z_2 = -z_1$. Hence, since g is odd,

$$g(z_1) = g(z_2) = g(-z_1) = -g(z_1)$$

and so $g(z_1) = 0$, $z_1 \neq 0$. However, from the above one sees that the only point in the unit disc where $g(z) = 0$ is $z = 0$. From this contradiction the univalency of $g(z)$ follows.

9. Considering Corollary 4.5.2 and Exercise 4.5, # 5 one arrives at the following result. If $f(z)$ is a univalent map of $\{1 < |z| < R_1\}$ onto $\{1 < |w| < R_2\}$ with the inner and outer boundaries corresponding to one another one can continue $f(z)$ into $\{R_1 < |z| < R_1^2\}$ by setting

$$f(z) = \frac{R_2^{\,2}}{\bar{f}(\varsigma)} \quad (\varsigma = \frac{R_1^{\,2}}{\bar{z}},\ R_1 \le |z| \le R_1^{\,2}).$$

In this way one can continue $f(z)$ to give a univalent mapping of $\{|z| > 0\}$ onto $\{|w| > 0\}$ with $f(z)$ being bounded near $z = 0$; in fact, $f(z) \to 0$ $(z \to 0)$. Hence $f(z)$ has a removable singularity at $z = 0$ and if we assume it has been removed then $f(z)$ is entire (Definition 3.6.1, p.115) and univalent.

By Picard's Theorem (Theorem 4.6.5, p.167 with $z_0 = \infty$) we see that

$f(z) = Az + B$ and since $f(0) = 0$ we must have that $f(z) = Az$. The unit circle is mapped onto the unit circle and hence $|A| = 1$. Consequently $R_2 = R_1$.

10. (Consider also Example 6.5.5, p.262)
From the point of view of conformal mapping the strip has two boundary points at ∞ according as we approach ∞ from the right or from the left. On the Riemann sphere there is one point at ∞, but in conformal mapping a point on the boundary in the set theoretic sense has to be considered together with how we approach the point from within the domain in question.

Let $z = z(w)$ map $\{\operatorname{Im} w > 0\}$ onto the strip so that $z(1) = +\infty$ and $z(-1) = -\infty$ where $+\infty$, $-\infty$ is ∞ approached from the right and left respectively. By the Riemann Mapping Theorem and Theorem 6.4.8 we know there is such a mapping $\varsigma(w)$ with ξ_1, ξ_2 on the real axis corresponding to $-\infty$, $+\infty$, i.e. $\varsigma(\xi_1) = -\infty$, $\varsigma(\xi_2) = +\infty$. Then for suitable real constants $A > 0$, B we have that

$$z(w) = \varsigma(Aw + B).$$

From the form of the Schwarz-Christoffel transformation

$$\frac{dz}{dw} = \frac{c}{1 - w^2}$$

for some constant C. Hence

$$z = \frac{c}{2} \log \frac{1 + w}{1 - w} + d.$$

Considering that as w traverses a small semi-circle from $1 - \varepsilon$ to $1 + \varepsilon$ in $\{\operatorname{Im} w > 0\}$ $\operatorname{Im} z$ increases from 0 to π we see that $c = 2$. Since the strip is invariant under lateral translations, apart from being real d is arbitrary.

Taking d = 0, then

$$e^z = \frac{1 + w}{1 - w}, \quad w = \frac{e^z - 1}{e^z + 1}.$$

Also

$$\zeta = \frac{w - i}{w + i}$$

is a univalent mapping of $\{\text{Im } w > 0\}$ onto $\{|\zeta| < 1\}$. Consequently

$$\zeta = \frac{(e^z - 1) - i(e^z + 1)}{(e^z - 1) + i(e^z + 1)} = f(z)$$

maps the strip onto $\{|\zeta| < 1\}$.

To say that g is continuous on the boundary of the strip means that $g(x)$, $g(x + i\pi)$ are continuous for $-\infty < x < \infty$ and that

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} g(x + i\pi) = g(+\infty),$$

$$\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} g(x + i\pi) = g(-\infty),$$

where $g(+\infty)$, $g(-\infty)$ are finite. If $z = F(\zeta)$ is the inverse of $\zeta = f(z)$ then $g(z) = g(F(\zeta)) = G(\zeta)$ is the function on $\{|\zeta| = 1\}$ corresponding to $g(z)$ on the boundary of the strip. Let $u(\zeta)$ be the solution of the Dirichlet problem for $\{|\zeta| < 1\}$ with boundary values $G(\zeta)$. When $u(f(z)) = U(z)$ is the solution of the Dirichlet problem for the strip with boundary values $g(z)$.

11. One can adapt the result of Example 6.5.4 or one can argue as follows. The (half) strip has 3 boundary points - $0, \pi, \infty$. From the Schwarz-Christoffel theory one know that $\{\text{Im } w > 0\}$ be mapped onto the half-strip so that a, b, c on the real w-axis correspond to $0, \pi, \infty$ for some a, b, c. By supplementing with a Mobius transformation one can assume that a, b, c are $0, 1, \infty$. Hence the transformation required is of the form

$$\frac{dz}{dw} = \frac{A}{w^{1/2}(1 - w)^{1/2}}.$$

The segment $[0, 1]$ of the real w-axis maps onto the segment $[0, \pi]$ of the real z-axis and hence $\frac{dz}{dw}$ is real for $0 < w < 1$ and so if we take $w^{1/2}\ (1 - w)^{1/2}$ to be positive for $0 < w < 1$, then we see that A is real (and positive). Integrating

$$\frac{dz}{dw} = \frac{A}{(w(1 - w))^{1/2}} = \frac{A}{((\frac{1}{2})^2 - (w - \frac{1}{2})^2)^{1/2}}$$

we obtain

$$z = A \sin^{-1}\left(\frac{w - 1/2}{1/2}\right) + B.$$

Now

$$\sin^{-1}\left(\frac{w - 1/2}{1/2}\right) = \begin{cases} -\frac{\pi}{2} & (w = 0) \\ \frac{\pi}{2} & (w = 1) \end{cases}$$

and so

$$z = \sin^{-1} \frac{w - 1/2}{1/2} + \frac{\pi}{2}$$

or

$$\sin(z - \frac{\pi}{2}) = \frac{w - 1/2}{1/2},$$

$$\frac{1}{2} - w = \frac{1}{2} \cos z.$$

Hence

$$w = \frac{1}{2}(1 - \cos z) = \sin^2 \frac{z}{2}.$$

Chapter 7

Ordinary Differential Equation

| READ §7.1 |

---

p.281 It is logically preferable, I think , to consider for $0 \le r < R$,

$$\sum_{n=0}^{\infty} \frac{r^n}{R^n} \cos n(\theta - \phi) = \mathrm{Re}\Big[\sum_{n=0}^{\infty} \frac{r^n}{R^n} (\cos n(\theta - \phi) + i \sin n(\theta - \phi))\Big]$$

$$= \mathrm{Re}\Big[\sum_{n=0}^{\infty} \Big(\frac{r}{R} e^{i(\theta - \phi)}\Big)^n\Big].$$

The rather abrupt introduction of $\frac{\zeta + z}{\zeta - z}$ is to some extent 'putting the cart before the horse'.

p.282 Note that $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, $z = \rho \cos \phi$ and so we can find the derivatives of u with respect to ρ, ϕ, θ in terms of $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$ and so on. Hence Laplace's equation in spherical co-ordinates can be verified.

Self-Assessment Questions

SAQ 1. Exercises 7.1, #1.

SAQ 2. Exercises 7.1, #2.

SAQ 3. Verify that the Legendre polynomial

$$w(z) = P_n(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n} [(z^2 - 1)^n]$$

satisfies the Legendre equation

$$(1 - z^2)W'' - 2zw' + n(n + 1)W = 0$$

[Hint: Put $v = (z^2 - 1)^n$ and show that

$(1 - z^2)v'' + 2(n - 1)zv' + 2nv = 0$ and differentiate n times.]

Preliminary discussion to §7.2

Given an ordinary differential equation and an initial condition it may not be clear if there is a solution to the problem or if there are several distinct ones. To cope with such problems we require existence and uniqueness results. The simplest equation is of order 1 and it is with such equations that Theorem 7.2.1 deals.

An equation of order 1 is of the form

$$\phi(x, y)\frac{dy}{dx} + \psi(x, y) = 0$$

or

$$\frac{dy}{dx} = f(x, y) \ (f(x, y) = -\frac{\psi(x, y)}{\phi(x, y)}).$$

One is assuming $\phi(x, y) \neq 0$ and this seems a reasonable assumption. The equation for Theorem 7.2.1 is this one in a different notation, i.e.

$$\dot{x}(t) = \frac{dx}{dt} = u(t, x).$$

A very simple equation of this form is when $u(t, x) = tx$ and then, separating the variables, $x(t) = Ae^{t^2/2}$ for some constant A. An initial condition $x(t_0) = x_0$ will determine A uniquely from $x_0 = Ae^{t_0^2/2}$.

What seems a simple equation is obtained by taking $u(t, x) = t^2 + x$. However, obtaining an explicit solution in this case is not at all trivial. If one assumes a solution of the form $x(t) = \sum_0^\infty a_k t^k$ one can determine the a_k and then the solution is

$$x(t) = -2(1 + t + \frac{t^2}{2}) + (2 + A)e^t,$$

for some constant A. An initial condition will determine A and so give a unique solution.

Taking $u(t, x) = t^2 + x^2$ leads to an equation that appears not to have any nice explicit solutions. However, proceeding as with the preceding equation one can show with some difficulty that a solution does exist. One sees that what seems a very nice function for $u(t, x)$ can lead to an equation which is not easy to handle. The result of Theorem 7.2.1 gives conditions on $u(t, x)$ that enable one to say, at any rate, that there is a solution and for given initial condition it is unique. It is necessary to impose some conditions on $u(t, x)$ so that the problem becomes manageable and those of Theorem 7.2.1 are weak enough so that for most applications the result is adequate.

Existence and uniqueness theorems are of intrinisic interest to the pure mathematician, but even for applications they can be useful. It may be possible to obtain information about some physical problem if one knows a unique solution exists, from the equation itself perhaps, without having any explicit form of this solution available.

In § 7.2 one finds that having coped with the first order equation one can extend the results to higher orders. Therefore Theorem 7.2.1 is of importance that extends beyond its immediate context.

| READ pp.284-287 |

Theorem 7.2.1 The proof proceeds by solving the integral equation

$$(*) \quad x(t) = x_0 + \int_{t_0}^{t} u[\tau, x(\tau)]d\tau$$

and showing that this equation is equivalent to the differential equation with initial condition. The integral equation has two advantages over the initial form of the problem:

(i) the initial condition $x(t_0) = x_0$ is built into (*);

(ii) it is much easier to estimate the effect of integration on a function than the effect of differentiation.

The approximation argument of the proof is quite 'natural' and you should avoid being confused by the detail. Note two essential requirements:

(i) at each stage one must check that $x_k(t)$ satisfies the condition at the top of p.285 so that $x_{k+1}(t)$ can be properly defined;

(ii) one must check that $x_k(t)$ converges to a limit function with the desired properties.

As far as (ii) is concerned, because the argument proceeds by successive approximations it is appropriate to think of the limit function as given by

$$x_0 + \sum_{n=1}^{\infty} (x_n(t) - x_{n-1}(t)).$$

p.287 The formula preceding Theorem 7.2.2 should begin $z = x + iy$ and not $\dot{z} = \dot{x} + i\dot{y}$.

| READ pp.287-296 |

---

p.288 The definitions and properties of functions analytic in two variables and associated double power series are dealt with earlier in § 4.7.

p.289 The integral equation at the top of the page is arrived at on the supposition that the differential equation of Theorem 7.2.3 can be solved. It is, therefore, an alternative form of the problem.

p.290 Note carefully the use of majorisation in the 'Second Proof' of Theorem 7.2.3.

Self-Assessment Questions

SAQ 4. Solve the equation

$$\dot{x}(t) = t^2 + x$$

with initial condition $x(0) = 1$.

SAQ 5. Show that for any prescribed initial condition,

$$\dot{x}(t) = t^2 + x^2$$

has an entire function solution.

| READ § 7.3 |

Definition 7.3.1 Note the linear dependence for functions requires $c_1w_1(z) + c_2w_2(z)$ being 0 for all z in $\{|z| < R\}$. It is essential that later w_1, w_2 are thought of as functions and not as special values.

Theorem 7.3.1 I find the proof of this theorem rather unconvincing and perhaps the following outline is more persuasive. If w_1, w_2 are linearly dependent, then for each z_0 in $\{|z| < R\}$ the pair of equations at the top of p.298 have a non-trivial solution and so the determinant of the system $W(z_0)$ is 0. Hence $W(z) \equiv 0$ in $\{|z| < R\}$.

Suppose next that $W(z) \equiv 0$ and that $w_1(z_0) \neq 0$ for some z_0 in $\{|z| < R\}$. There is a neighbourhood of z_0 in which $w_1(z) \neq 0$ and for such z

$$\frac{d}{dz}\left(\frac{w_2(z)}{w_1(z)}\right) = -\frac{W(z)}{w_1^{\,2}(z)} = 0.$$

In this neighbourhood, therefore, $w_2(z) = c.w_1(z)$ for some constant c. By analytic continuation this equation holds throughout $\{|z| < R\}$ and so w_1, w_2 are linearly dependent. If there is no such z_0, then $w_1(z) \equiv 0$ in $\{|z| < R\}$ and in this case linear dependent is trivial.

Problems for §7.1-§7.3

P1. Exercise 7.1, #3.

P2. Exercises 7.2, #1, excluding final part of question.

P3. Exercises 7,3, #1.

P4. Exercises 7.3, #2.

P5. Exercises 7.3, #5.

| READ §7.4 |

Theorem 7.4.1 It may not be clear in the proof where the condition $\text{Re}[f(0)] \geq 1$ is used, since this is not pointed out. From the recursive relations for c_k and C_k one observes that each coefficient will be properly defined if and only if $k + a_0 \neq 0$ for $k \geq 0$. As $a_0 = f(0)$, the condition $\text{Re}[f(0)] \geq 1$ ensures that $k + a_0 \neq 0$.

The condition $\text{Re}[f(0)] \geq 1$ is considered because of the discussion preceding the statement that $\text{Re}[f(0)] \neq -k$ for any non-negative integer k.

p.309 The Green's function considered here is rather different from those considered in the preceding chapter. However, in each case $G(z, \zeta)$ is a kernel which if integrated appropriately together with a given function gives a function of the kind sought. In the earlier chapter the function sought was a harmonic function with prescribed boundary behaviour and in this case a particular solution of the non-homogeneous differential equation.

Problems for §7.4

P.6, 7, 8, 9, 10, 11 are Exercises 7.4, #1, 2, 3, 4, 5, 6.

| READ §7.5 |

In this section factors like z^ν occur where ν may not be an integer and so z^ν may have different determinations. In such instances one should think of z^ν being positive for z small and positive. Other values are obtained by analytic continuation.

p.312 In the formula for $J_\nu(z)$, $\sum_{n=1}^{\infty}$ should be $\sum_{n=0}^{\infty}$.

p.312 The discussion about the linear independence of $J_\nu(z)$, $J_{-\nu}(z)$ when ν is half an odd positive integer is prompted by the fact that in this case $\nu - (-\nu) = 2\nu$, a positive integer. However, it may happen that the series method for solving a differential equation will yield linearly independent solutions even though the roots of the indicial equation differ by a positive integer and this is the case here.

Note also that from the form of Bessel's equation if $J_\nu(z)$ is a solution then so is $J_{-\nu}(z)$. When ν is half an odd positive integer then $J_\nu(z)$, $J_{-\nu}(z)$ behave differently near $z = 0$; one is bounded and the other is unbounded and so they must be linearly independent.

Problems for §7.5

P.12, 13, 14, 15, 16, 17 are Exercises 7.5, #1, 2, 3, 4, 5, 6. [In #5, y' should be y.]

| READ §7.6 |

---

pp.318, 319 The natural approach would be to find the form of $P_n(z)$ by considering a solution of the form

$$p(z) = a_0 + a_1 z + - - a_n z^n$$

and finding a recursive relation that specified each of $a_0, a_1, \ldots$ in terms of the preceding. However, we can also consider

$$q(z) = z^n(a_n + \frac{a_{n-1}}{z} + - - -) = z^n(b_0 + \frac{b_1}{z} + - - + \frac{b_n}{z^n})$$

and find each of $b_0, b_1, \ldots$ in terms of the preceding. For the Legendre polynomials, if one 'knows the answer' then one realises that is is better, i.e. simpler, to use $q(z)$ rather than $p(z)$.

Problems for §7.6

P.18, 19, 20, 21, 22 are Exercises 7.6, #1, 2, 4, 7, 8.

| READ pp.325-333, excluding Theorem 7.7.4 |

Sturm-Liouville theory has a number of important applications, but a rigorous account of the theory is rather difficult. Dettman's presentation has some drawbacks and I hope the following commentary will help you to cope with these.

p.327 et seq. The properties of the Wronskians that arise in this context are not adequately dealt with. If u, v are solutions of

$$y'' + [\lambda - q(x)]y = 0$$

then

$$\frac{d}{dx} W(u, v) = uv'' - u''v = -u.(\lambda - q)v + (\lambda - q)u.v = 0$$

and so $W(u, v)$ is constant, i.e. independent of x. This is observed at the top of p.329. Hence one sees that the result about the Wronskian and linear independence considered earlier is still true here.

Theorem 7.7.3 As far as proving that ϕ, ψ as defined in Theorem 7.7.2 satisfy the given integral equations one could do this by direct verifications and so avoid the iteration procedure. This procedure does seem necessary in order to prove that $W(\lambda)$ is an entire function of λ.

p.331 Near the top of the page the claim that $\phi(x, \lambda)$, $\psi(x, \lambda)$ being entire implies that $W(\lambda)$ is entire is not quite correct since $W(\lambda)$ also involves $\phi'(x, \lambda)$, $\psi'(x, \lambda)$. From the differential equation one sees that ϕ'', ψ'' are entire in λ since ϕ, ψ are and hence it follows that ϕ', ψ' are.

p.332 The reference to 'the Fourier series of f(x)' near the bottom of the page is a bit misleading. The series arising in Sturm-Liouville theory are a generalisation of Fourier series involving sines and cosines.

Theorem 7.7.4 In the proof of the theorem note that $\cos \varsigma(x - a)$ means $\cos[\varsigma(x - a)]$ and not $(x - a).\cos \varsigma$.

You may not find the formulas at the beginning of the proof as easy to establish as the author assumes. The first of these can be obtained as follows.

Let $\phi(t, \lambda)$ be the solution of $\phi'' + (\lambda - q)\phi = 0$ with $\phi(a) = \sin\alpha$, $\phi'(a) = -\cos\alpha$ and $\lambda = \varsigma^2$. Then, with $\phi(t) = \phi(t, \varsigma^2)$,

$$\int_a^x \sin\{\varsigma(x - t)\}q(t)\,\phi(t)dt = \int_a^x \sin\{\varsigma(x - t)\}[\phi''(t) + \varsigma^2\phi(t)]dt$$

and

$$\begin{aligned}\int_a^x \sin\{\varsigma(x - t)\}\phi''(t)dt &= \sin\{\varsigma(x - t)\}\phi'(t)\Big|_a^x + \varsigma\int_a^x \cos\{\varsigma(x - t)\}\phi'(t)dt\\ &= -\sin\{\varsigma(x - a)\}\phi'(a) + \varsigma\cos\{\varsigma(x - t)\}\phi(t)\Big|_a^x\\ &\qquad - \varsigma^2\int_a^x \sin\{\varsigma(x - t)\}\phi(t)dt\\ &= \cos\alpha.\sin\{\varsigma(x-a)\} + \varsigma[\phi(x) - \sin\alpha.\cos\{\varsigma(x-a)\}]\\ &\qquad - \varsigma^2\int_a^x \sin\{\varsigma(x - t)\}\phi(t)dt.\end{aligned}$$

Using this result in the initial equation gives the formula for $\phi(x, \varsigma^2)$.

p.335 Figure 7.7.1 is relevant to the particular case $a = 0$, $b = 1$. For given parameters a, b one has to replace $(n + \frac{1}{2})\pi$ by $(n + \frac{1}{2})\frac{\pi}{b - a}$.

p.336 The verification that the formula at the bottom of the page follows from the preceding is quite lengthy. One changes the order of integration and uses the calculus of residues on each of the integrands in $\int_a^x$ and $\int_x^b$. If these two integrals are then evaluated their contributions fit together nicely to give the final formula.

p.337 The concluding sentence of the proof of Theorem 7.7.4 may leave you confused. The discussion preceding the statement of Theorem 7.7.4 shows that $\lim_{n\to\infty} \frac{1}{2\pi i} \int_{C_n} y(x, \lambda)d\lambda$ is expressible as a series with respect to the eigenfunctions. In the proof of Theorem 7.7.4 it is shown that this limit is also expressible as Fourier series. Using results about the convergence of Fourier series from §8 one establishes the result of the theorem.

p.338 The boundary conditions in Corollary 7.7.4 are

$$\cos\alpha \,.\, y(a) + \sin\alpha . y'(a) = 0,$$
$$\cos\beta \,.\, y(b) + \sin\beta . y'(b) = 0,$$

as in (a) on p.326. Noting on p.332 that $\psi(x, \lambda_k) = A_k\, \phi(x, \lambda_k)$ and on p.333 that $\Phi_k(x) = [\frac{A_k}{w'(\lambda_k)}]^{1/2}\, \phi(x, \lambda_k)$, then, since $\phi(a, \lambda_k) = \sin\alpha$, $\phi'(a, \lambda_k) = -\cos\alpha$ and $\psi(b, \lambda_k) = \sin\beta$, $\psi'(b, \lambda_k) = -\cos\beta$, it follows that the integrated term on line 3 from the top is 0 at both upper and lower limits.

Problems for §7.7

P.23, 24 are Exercise 7.7, #1, 2.

| READ §7.8 |

If one assumes that G has properties 1, 2, 3, 4 on p.340, then one can prove directly that the solution of $(py')' - qy = -f$ satisfying the boundary conditions

$$\cos\alpha \,.\, y(a) + \sin\alpha . y'(a) = 0,$$
$$\cos\beta \,.\, y(b) + \sin\beta . y'(b) = 0,$$

is given by

$$y(x) = \int_a^b G(x, t)\, f(t)dt.$$

Consider $G = G(x, t)$ with x fixed in (a, b) as a function of t and $G' = G'(t) = \frac{\partial}{\partial t} G(x, t)$. Then

$$\int_a^b G(py')'dt = \left(\int_a^x + \int_x^b\right) G(py')'dt$$

and, thinking of $\int_a^x = \lim_{\xi \to x} \int_a^\xi$ with $\xi < x$,

$$\int_a^x G(py')' = Gpy'\Big|_a^x - \int_a^x G'.py'$$

$$= p(x)y'(x)\, G(x, x^-) - p(a)y'(a)\, G(x, a) - \int_a^x (pG').y';$$

$$\int_a^x (pG')y' = pG'.y\Big|_a^x - \int_a^x (pG')'y$$

$$= p(x)\, G'(x, x^-)\, y(x) - p(a)G'(a)y(a) - \int_a^x qGy,$$

from 1. Hence,

$$-\int_a^x Gf = \int_a^x G[(py')' - qy] = -p(x)y(x)\, G'(x, x^-) + p(x)y'(x)\, G(x, x^-)$$

$$+ p(a)[y(a)G'(x, a) - y'(a)G(x, a)]$$

$$= -p(x)y(x)G'(x, x^-) + p(x)y'(x)G(x, x^-)$$

since the terms in the square bracket give 0 by 3, and the boundary conditions on y.

Similarly, thinking of $\int_x^b = \lim_{\eta \to x} \int_\eta^b$ with $\eta > x$,

$$-\int_x^b Gf = \int_x^b G[(py')' - qy] = p(x)y(x)G'(x, x^+) - p(x)y'(x)G(x, x^+)$$

and so

$$-\int_a^b Gf = y(x)p(x)[G'(x,x^+) - G'(x,x^-)] - p(x)y'(x)[G(x,x^+) - G(x,x^-)]$$

$$= y(x)$$

since $G(x, x^+) = G(x, x^-)$, by 2; and $G'(x, x^+) - G'(x, x^-) = -\frac{1}{p(x)}$, by 4.

Note that the Green's function is symmetric.

Problems for §7.8

P.25, 26 are Exercises 7.8, #1, 2.

Solutions to Self-Assessment Questions

1. If $y(x) = x^m$ is a solution then

$$x^2.m(m-1).x^{m-2} + ax.mx^{m-1} + bx^m = 0,$$

i.e.

$$(m(m-1) + am + b)x^m = 0$$

and so $m(m-1) + am + b = 0$. Conversely if m satisfies this quadratic equation then $y(x) = x^m$ is a solution.

2. Consider the argument at the top of p.283 and note that the Helmholtz equation differs from that of Laplace by the term λu. One then sees that the differential equation for $f(\rho)$ is that given.

If $x = \lambda^{1/2}\rho$, $y = \rho^{1/2}f$ and y', y",... denote derivatives with respect to x; f', f",... denote derivatives with respect to ρ, then

$$y' = \frac{d}{d\rho}(\rho^{1/2}f).\frac{d\rho}{dx} \qquad y'' = \frac{dy'}{d\rho}.\frac{d\rho}{dx}$$

$$= (\tfrac{1}{2}\rho^{-1/2}f + \rho^{1/2}f')\lambda^{-1/2}, \qquad = (-\tfrac{1}{4}\rho^{-3/2}f + \rho^{-1/2}f' + \rho^{1/2}f'')\lambda^{-1}.$$

Hence

$$x^2y'' + xy' + x^2 - ((n+\tfrac{1}{2})^2)y = \lambda\rho^2(-\tfrac{1}{4}\rho^{-3/2}f + \rho^{-1/2}f' + \rho^{1/2}f'').\lambda^{-1}$$

$$+ \lambda^{1/2}\rho(\tfrac{1}{2}\rho^{-1/2}f + \rho^{1/2}f')\lambda^{-1/2} + (\lambda\rho^2 - (n+\tfrac{1}{2})^2)\rho^{1/2}f$$

$$= \rho^{1/2}[\rho^2f'' + 2\rho f' + (\lambda\rho^2 - n(n+1))f]$$

and so result follows.

[Note: When proving A = B always consider whether it is simpler to start with A or to start with B.]

3. If $v = (z^2-1)^n$, then

$$v' = 2nz(z^2-1)^{n-1}$$

so that

$$(1-z^2)v' + 2nzv = 0.$$

Hence $(1-z^2)v'' + 2(n-1)zv' + 2nv = 0.$

Now differentiating n times and using Leibniz' rule,

$$[(1 - z^2)v^{(n+2)} - 2nzv^{(n+1)} - n(n - 1)v^{(n)}]$$
$$+ 2(n - 1)[zv^{(n+1)} + nv^{(n)}] + 2nv^{(n)} = 0$$

and therefore

$$(1 - z^2)v^{(n+2)} - 2zv^{(n+1)} + n(n + 1)v^{(n)} = 0,$$

i.e.

$$(1 - z^2) \frac{d^2}{dz^2}[v^{(n)}] - 2z \frac{d}{dz}[v^{(n)}] + n(n + 1) v^{(n)} = 0.$$

Therefore $v^{(n)}$ and so $\frac{n}{2^n n!} v^{(n)} = P_n$ satisfies the Legendre equation.

4. Assume that there is a solution of the form $x(t) = 1 + \sum_1^\infty a_k t^k$. Then

$$\sum_1^\infty ka_k t^{k-1} = t^2 + \sum_0^\infty a_k t^k \qquad (a_0 = 1)$$

or

$$\sum_0^\infty ((k + 1) a_{k+1} - a_k)t^k = t^2.$$

Hence

$$(k + 1)a_{k+1} - a_k = 0 \quad (k \neq 2)$$
$$= 1 \quad (k = 2),$$

and so

$$a_1 = a_0 = 1,\ 2a_2 = a_1,\ 3a_3 = a_2 + 1,\ 4a_4 = a_3, \ldots$$

Therefore,

$$a_0 = a_1 = 1,\ a_2 = \frac{1}{2},\ a_3 = \frac{3}{2.3},\ a_4 = \frac{3}{2.3.4}, \ldots$$

and consequently

$$x(t) = 1 + t + \frac{t^2}{2} + 3(\frac{t^2}{3!} + \frac{t^4}{4!} + \ldots)$$
$$= 1 + t + \frac{t^2}{2} + 3[1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots - (1 + t + \frac{t^2}{2})]$$
$$= 3e^t - 2(1 + t + \frac{t^2}{2}).$$

Check:

$$\dot{x}(t) = 3e^t - 2(1 + t),$$

$$t^2 + x(t) = t^2 + 3e^t - [2(1 + t) + t^2] = 3e^t - 2(1 + t).$$

5. In the form of Theorem 7.2.3 the equation is

$$w' = f(z, w) = z^2 + w^2 \qquad (t = z,\ x(t) = w(z)).$$

Since $z^2 + w^2$ is analytic in z, w for all z, w the result follows from Theorem 7.2.3.

If one were to argue directly as in the 'Second Proof' on p.290 then we are led to consider ρ_1, ρ_2 so that the coefficients of z^2, w^2 in the series for $\dfrac{M\rho_1\rho_2}{(\rho_1 - z)(\rho_2 - w)}$ $(|z| < \rho_1,\ |w| < \rho_2)$ are bounded by 1. Now

$$\frac{M\rho_1\rho_2}{(\rho_1 - z)(\rho_2 - z)} = \frac{M}{(1 - z/\rho_1)(1 - w/\rho_2)}$$

$$= M\left(1 + \frac{z}{\rho_1} + \frac{z^2}{\rho_1^2} + \ldots\right)\left(1 + \frac{w}{\rho_2} + \frac{w^2}{\rho_1^2} + \ldots\right)$$

$$= M\left(1 + \frac{z}{\rho_1} + \frac{w}{\rho_2} + \frac{z^2}{\rho_1^2} + \frac{w^2}{\rho_2^2} + \frac{zw}{\rho_1\rho_2} + \ldots\right)$$

and given any $\rho_1, \rho_2 > 0$ we can choose M large enough so that $\dfrac{M}{\rho_1^2}, \dfrac{M}{\rho_2^2} \geq 1$.

Hence it follows that the solution of the problem is analytic for all t, i.e. it is entire in t.

Solutions to Problems

1. Differentiate Legendre's equation

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n + 1)y = 0$$

m times with respect to x and put $v = \dfrac{d^my}{dx^m}$ and we obtain

$$(1 - x^2)\frac{d^2v}{dx^2} - 2x(m + 1)\frac{dv}{dx} + (n - m)(n + m + 1)v = 0.$$

Now set $w = (1 - x)^{m/2}v$ and express the preceding equation in terms of w to obtain

$$(1 - x^2)\frac{d^2w}{dx^2} - 2x\frac{dw}{dx} + \left[n(n + 1) - \frac{m^2}{1 - x^2}\right]w = 0.$$

2. Put $x(t) = \dot{y}(t)$ and consider first the equation

$$\dot{x}(t) = -f(t) \quad (t \geq 0)$$

with initial condition $x(0) = b$. From Theorem 7.2.1 with $u(t, x) = -f(t)$ there is a unique solution

$$x(t) = b - \int_0^t f(\tau)d\tau .$$

Next consider the equation

$$\dot{y}(t) = b - \int_0^t f(\tau)d\tau \quad (t \geq 0)$$

with initial condition $y(0) = a$. From Theorem 7.2.1 with $u(t, y) = b - \int_0^t f(\tau)d\tau$ there is a unique solution

$$y(t) = a + \int_0^t (b - \int_0^\tau f(u)du)d\tau$$

$$= a + bt - \int_0^t (\int_0^\tau f(u)du)d\tau .$$

Interchanging the order of integration in the repeated integral we find that

$$\int_0^t (\int_0^\tau f(u)du)d\tau = \int_0^t (t - u)f(u)du$$

and hence result follows.

3. Let $w(z) = w(\frac{1}{\zeta}) = W(\zeta)$. Then

$$w'(z) = W'(\zeta).\frac{d\zeta}{dz} = W'(\zeta).(-\frac{1}{z^2}) = -\zeta^2 W(\zeta);$$

$$w''(z) = -(2\zeta W'(\zeta) + \zeta^2 W''(\zeta))\frac{d\zeta}{dz}$$

$$= \zeta^3(2W'(\zeta) + \zeta W''(\zeta)).$$

Put $p(\frac{1}{\zeta}) = P(\zeta)$, $q(\frac{1}{\zeta}) = Q(\zeta)$, $r(\frac{1}{\zeta}) = R(\zeta)$ and the equation becomes

$$\zeta^3(2W'(\zeta) + \zeta W''(\zeta)) + P(\zeta).(-\zeta^2 W'(\zeta)) + Q(\zeta)\ W(\zeta) = R(\zeta),$$

i.e.

$$W''(\zeta) + (\frac{2}{\zeta} - \frac{P(\zeta)}{\zeta^2})\ W'(\zeta) + \frac{Q(\zeta)}{\zeta^4}\ W(\zeta) = \frac{R(\zeta)}{\zeta^4}.$$

From the definition of ordinary point the preceding equation will have $\zeta = 0$ as an ordinary point if and only if each of

$$\frac{2}{\varsigma} - \frac{P(\varsigma)}{\varsigma^2}, \frac{Q(\varsigma)}{\varsigma^4}, \frac{R(\varsigma)}{\varsigma^4}$$

are analytic at $\varsigma = 0$, i.e. $\varsigma = 0$ is a removable singularity of each of these functions. Equivalently, if and only if each of

$$2z - z^2p(z), z^4q(z), z^4r(z)$$

is analytic at $z = \infty$. This means that for large z,

$$p(z) = \frac{2}{z} + \frac{a_{-2}}{z^2} + \frac{a_{-3}}{z^3} + \ldots,$$

$$q(z) = \frac{b_{-4}}{z^4} + \frac{b_{-5}}{z^5} + \ldots,$$

$$r(z) = \frac{c_{-4}}{z^4} + \frac{c_{-5}}{z^5} + \ldots,$$

where the power series in $\frac{1}{z}$ converge for all large z.

The equation for $W(\varsigma)$ has a solution for small ς,

$$W(\varsigma) = c_0 + c_1\varsigma + c_2\varsigma^2 + \ldots$$

assuming that $\varsigma = 0$ is an ordinary point. This is equivalent to saying that if $z = \infty$ is an ordinary point of the original equation then this equation has a solution

$$w(z) = c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \ldots$$

which is valid for large z.

4. The equation has solutions analytic in $\{|z| < 1\}$ from the earlier discussion in the section. Let $w(z) = \sum_0^\infty a_k z^k$ and then

$$(1 - z^2) \sum_2^\infty k(k - 1)a_k z^{k-2} - 2z \sum_1^\infty k a_k z^{k-1} + n(n + 1) \sum_0^\infty a_k z^k = 0.$$

The coefficient of z^k from the individual contributions on the left is equated to 0 gives us

$$(k + 2)(k + 1)a_{k+2} - k(k - 1)a_k - 2ka_k + n(n + 1)a_k = 0 \quad (k \geq 0)$$

and so

$$\begin{aligned}(k + 2)(k + 1)a_{k+2} &= -[n(n + 1) - k(k + 1)]a_k \\ &= -(n - k)(n + k + 1)a_k,\end{aligned}$$

i.e.

$$a_{k+2} = -\frac{(n-k)(n+k+1)}{(k+2)(k+1)} a_k \quad (k \geq 0).$$

Two independent solutions are obtained by taking a_0, a_1 to be 0, 1 and 1, 0 respectively.

If n is a positive integer then $n - k = 0$ for $k = n$ and then a_{n+2}, $a_{n+4}, \ldots$ are all 0 and so we get a polynomial.

Remark: If one assumes a power series solution then one sees what the form of this must be and one can check directly that there are two independent solutions analytic in $\{|z| < 1\}$.

5. On p.298 we see that

$$W(z) = W_0 \exp\left[-\int_{z_0}^{z} p(\zeta)d\zeta\right].$$

If we continue $W_1(z)$, $W_2(z)$ round C to obtain $\omega_1(z)$, $\omega_2(z)$ then the effect on W(z) is found by continuing $\int_{z_0}^{z} p(\zeta)d\zeta$ round C. Denoting by $\left[\int_{z_0}^{z} p(\zeta)d\zeta\right]_1$ the value before continuation and $\left[\int_{z_0}^{z} p(\zeta)d\zeta\right]_2$ the value afterwards we obtain

$$\left[\int_{z_0}^{z} p(\zeta)d\zeta\right]_2 - \left[\int_{z_0}^{z} p(\zeta)d\zeta\right]_1 = \int_C p(\zeta)d\zeta = 2\pi iS.$$

Hence, in the notation of the question,

$$\begin{aligned}W(\omega_1, \omega_2) &= W_0 \exp\left(-\left[\int_{z_0}^{z} p(\zeta)d\zeta\right]_2\right)\\ &= W_0 \exp\left(-\left[\int_{z_0}^{z} p(\zeta)d\zeta\right]_1 - 2\pi iS\right)\\ &= W_0 \exp\left(-\left[\int_{z_0}^{z} p(\alpha)d\zeta\right]_1\right).\exp(-2\pi iS)\\ &= W(w_1, w_2)e^{-2\pi iS}.\end{aligned}$$

6. If we set $w(z) = w(\frac{1}{\zeta}) = W(\zeta)$ as in the solution to P.3 then from that solution we see that

$$W''(\zeta) + \left(\frac{2}{\zeta} - \frac{P(\zeta)}{\zeta^2}\right)W'(\zeta) + \frac{Q(\zeta)}{\zeta^4} = 0,$$

where $P(\zeta) = p(\frac{1}{\zeta})$, $Q(\zeta) = q(\frac{1}{\zeta})$. The original equation will have a regular singular point at $z = \infty$ if the above equation has one at $\zeta = 0$. Multiplying by ζ^2 this will be the case if $2\zeta - P(\zeta)$ has a zero at $\zeta = 0$ and $\frac{Q(\zeta)}{\zeta^2}$ is analytic at $\zeta = 0$. Hence we require that for ζ near $\zeta = 0$,

$$P(\zeta) = a_1\zeta + a_2\zeta^2 + \ldots,\quad Q(\zeta) = b_2\zeta^2 + b_3\zeta^2 + \ldots\ .$$

Therefore the original equation will have a regular singular point at $z = \infty$ if for large z,

$$p(z) = \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots,\quad q(z) = \frac{b_2}{z^2} + \frac{b_3}{b^3} + \ldots\ .$$

7. Consider Theorem 7.4.1 and the preceding discussion. We shall get formal solutions since the procedure stated will amount to solving

$$zu'' + f(z)u' + g(z)u = 0$$

as in Theorem 7.4.1, where f, g depend on $m = m_1, m_2$, the roots of

$$m(m - 1) + p(0)m + q(0) = 0.$$

If $m_1 - m_2$ is not an integer, then $z^{m_1}\phi(z)$, $z^{m_2}\psi(z)$, where ϕ, ψ are analytic at $z = 0$ and neither is identically 0, cannot have the same behaviour at $z = 0$. Note that if $\phi(z) = Az^M + \ldots$, $\psi(z) = Bz^N + \ldots$, then $M + m_1 \neq N + m_2$.

8. Consider the equation

$$w'' + p(z)w' + q(z) = 0.$$

This equation has $z_0 \in C$ as an ordinary point if p(z), q(z) are analytic at z_0. Hence if every point z_0 in the complex plane is an ordinary point then p(z), q(z) are analytic for all complex z, i.e. they are entire.

From the solution to P.6 if $z = \infty$ is a regular singular point, then $p(z) \to 0$, $q(z) \to 0$ as $z \to \infty$. Hence p(z), q(z) are bounded and so constant by Liouville's theorem. Clearly the constant value in each case is 0 and so the equation reduces to $w'' = 0$.

[Remark: We use that if f(z) is entire and $f(z) \to 0$ $(z \to \infty)$ then $f(z) \equiv 0$. This is proved as follows.

To say $f(z) \to 0$ $(z \to \infty)$ means that given $\varepsilon > 0$ there is an M such that

$$|f(z)| < \varepsilon \quad (|z| \geq M).$$

Since $|f(z)| < \varepsilon$ $(|z| = M)$ it follows that $|f(z)| < \varepsilon$ $(|z| < M)$ by the Maximum Modulus Principle (Theorem 3.7.1, p.119). Hence f(z) is bounded and so it is constant by Liouville's theorem (Theorem 3.6.4, p.115). Clearly $f(z) \equiv 0$.]

9. When all points are ordinary apart from $z = 0, \infty$, then

$$w'' + p(z)w' + q(z)w = 0,$$

where p(z), q(z) are analytic for all complex z other than $z = 0$. If $z = 0$ is a regular singular point, then $z^2p(z)$ has a zero at $z = 0$ and $z^2q(z)$ is analytic at $z = 0$; and if $z = \infty$ is also a regular singular point, then zp(z) and $z^2q(z)$ are bounded near $z = 0$, by solution to P.6. Therefore zp(z), $z^2q(z)$ are entire and bounded and hence constant by Liouville's theorem. This means that $p(z) = \frac{A}{z}$, $q(z) = \frac{B}{z^2}$ for some constants A, B and then the equation is

$$z^2w'' + Am + B = 0.$$

If $z = \frac{1}{\zeta}$, $w(z) = W(\zeta)$ then the equation comes

$$\zeta^2W'' + (2 - A)\zeta W' + BW = 0,$$

from solution to P.3. Hence the indicial equation at $z = \infty$, $\zeta = 0$ is

$$n(n - 1) + (2 - A)n + B = 0.$$

Therefore

$$m_1 + m_2 = 1 - A, \qquad n_1 + n_2 = A - 1,$$

$$m_1m_2 = B, \qquad n_1n_2 = B.$$

and so

$$m_1 + m_2 + n_1 + n_2 = 0, \qquad m_1m_2 = n_1n_2.$$

10. If all points are ordinary apart from 0, 1, ∞ which are regular singular then

$$w'' + p(z)w' + q(z) = 0$$

where p, q are analytic in the plane apart from $z = 0, 1$. When $z = 0, 1$ are regular singular then p(z) has at most simple poles at these points and q(z) has at most double poles at these points. Therefore

$$p(z) = \frac{A}{z} + \frac{B}{z - 1} + \phi(z),$$

$$q(z) = \frac{C}{z} + \frac{D}{z^2} + \frac{E}{(z-1)} + \frac{F}{(z-1)^2} + \psi(z),$$

where $\phi(z)$, $\psi(z)$ are entire. If $z = \infty$ is a regular singular point then $zp(z)$, $z^2q(z)$ is bounded near $z = \infty$ and so are $z\phi(z)$, $z\psi(z)$. Consequently $\phi(z)$, $\psi(z)$ are bounded and $\phi(z)$, $\psi(z) \to 0$ $(z \to \infty)$ so that $\phi(z) = 0$, $\psi(z) \equiv 0$. Also $z^2(\frac{C}{z} + \frac{E}{z-1})$ is bounded near $z = \infty$ so that $E = -C$. Hence the equation is

$$W'' + (\frac{A}{z} + \frac{B}{z-1})\, W' + \{\frac{D}{z^2} + \frac{E}{(z-1)^2} + C(\frac{1}{z} - \frac{1}{z-1})\}\, W = 0,$$

and this is of the form given in the question.

Consider then the equation of the question. The discussion on p.303 and the form of the equation at $z = 0, 1, \infty$ shows that

$m(m - 1) + a_0m + b_0 = 0$ has roots α_1, α_2;

$m(m - 1) + a_1m + b_1 = 0$ has roots β_1, β_2;

and, using the equation transformed by $z = \frac{1}{\zeta}$,

$m(m - 1) + (2 - a_0 - a_1)m + (b_0 + b_1 - c_0) = 0$ has roots γ_1, γ_2.

Hence

$$\alpha_1 + \alpha_2 = 1 - a_0,\ \beta_1 + \beta_2 = 1 - a_1,$$

$$\alpha_1\alpha_2 = b_0,\ \beta_1\beta_2 = b_1,$$

$$\gamma_1 + \gamma_2 = a_0 + a_1 - 1,\ \gamma_1\gamma_2 = b_0 + b_1 - c_0.$$

and so

$$c_0 = b_0 + b_1 - \gamma_1\gamma_2 = \alpha_1\alpha_2 + \beta_1\beta_2 - \gamma_1\gamma_2,$$

$$\alpha_1 + \alpha_2 + \beta_1 + \gamma_1 + \gamma_2 = 1 - a_0 + 1 - a_1 + a_0 + a_1 - 1 = 1,$$

P.11. Take $w(z) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha + b)\Gamma(\beta + k)}{\Gamma(\gamma + k)} \frac{z^k}{k!}$ in the left hand side of the equation and we get

$$z(1-z) \sum_{k=2}^{\infty} \frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(\gamma+k)} \frac{z^{k-2}}{(k-2)!} + [\gamma-(\alpha+\beta+1)z] \sum_{1}^{\infty} \frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(\gamma+k)} \frac{z^{k-1}}{(k-1)!}$$

$$- \alpha\beta \sum_{0}^{\infty} \frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(\gamma+k)} \frac{z^k}{k!},$$

and the coefficient of z^k is

$$A_k = \frac{\Gamma(\alpha+k+1)\Gamma(\beta+k+1)}{\Gamma(\gamma+k+1)}.\frac{1}{(k-1)!} - \frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(\gamma+k)}.\frac{1}{(k-2)!} + \gamma\,\frac{\Gamma(\alpha+k+1)\Gamma(\beta+k+1)}{\Gamma(\gamma+k+1)}\,\frac{1}{k!}$$

$$- (\alpha+\beta+1)\,\frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(\gamma+k)}.\frac{1}{(k-1)!} - \alpha\beta\,\frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(\gamma+k)}.\frac{1}{k!}.$$

We set $\frac{1}{n!} = 0$ whenever n is a negative integer. Hence

$$A_k = \frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(\gamma+k).k!}\,[\frac{(\alpha+k)(\beta+k)}{\gamma+k}.k - k(k-1) + \gamma\,\frac{(\alpha+k)(\beta+k)}{\gamma+k}$$

$$- (\alpha+\beta+1).k - \alpha\beta\,]$$

$$= \frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(\gamma+k)}\,[\frac{(\alpha+k)(\beta+k)}{\gamma+k}\,(k+\gamma) - \{k^2 + (\alpha+\beta)k + \alpha\beta\}]$$

$$= \frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(\gamma+k)}\,[(\alpha+k)(\beta+k) - (k+\alpha)(k+\beta)]$$

$$= 0$$

Hence $F(\alpha, \beta, \gamma, z)$ satisfies the equation.

One can verify in the same tedious way that the other function is also a solution.

The functions have factors $\Gamma(\gamma + k)$ and $\Gamma(2 - \gamma + k)$ and the condition k not an integer ensures these are non-zero,

12. In the section it is proved that $J_\nu(z)$, $J_{-\nu}(z)$ are linearly dependent if and only if ν is an integer. In this case $W(z) \equiv 0$ and $-\frac{2 \sin \nu\pi}{z} \equiv 0$ and hence the result is true in this case.

Assuming ν is not an integer, first of all we argue as in the proof of Theorem 7.3.2 to obtain

$$W(z) = \frac{A}{z}$$

for some constant A. Considering the initial terms of the series for $J_\nu(z)$, $J_{-\nu}(z)$ on p.312 (remember the sum for $J_\nu(z)$ starts at n = 0 and not at n = 1) we see that

$$J_\nu(z) = \frac{(1/2z)^\nu}{\Gamma(1+\nu)} + \ldots,\quad J'_\nu(z) = \frac{\nu}{2}\,\frac{(1/2z)^{\nu-1}}{\Gamma(1+\nu)} + \ldots,$$

and so

$$W(z) = -\frac{\nu/2.(1/2z)^{-1}}{\Gamma(1+\nu)\Gamma(1-\nu)} - [\frac{\nu/2(1/2z)^{-1}}{\Gamma(1-\nu)\Gamma(1+\nu)}] + \ldots$$

$$= -\frac{2\nu}{\Gamma(1+\nu)\Gamma(1-\nu)}.\frac{1}{z} + \ldots,$$

and hence

$$W(z) = -\frac{2\nu}{\Gamma(1+\nu)\Gamma(1-\nu)}.\frac{1}{z} = -\frac{2}{\Gamma(\nu)\Gamma(1-\nu)}.\frac{1}{z}.$$

On p.197 it is proved that

$$\Gamma(z)\Gamma(1-z) = \pi \operatorname{cosec} \pi z = \frac{\pi}{\sin \pi z}$$

and taking $z = \nu$ we obtain

$$W(z) = -\frac{2 \sin \nu\pi}{\pi z}.$$

13. Putting $\nu = \frac{1}{2}$ in the sum for $J_\nu(z)$ we get

$$J_{1/2}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (1/2z)^{2n+1/2}}{n!\Gamma(1/2 + n + 1)}$$

and so

$$(\frac{z}{2})^{1/2} J_{1/2}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (1/2z)^{2n+1}}{n!\Gamma(n + 3/2)}.$$

Now

$$\Gamma(n + \frac{3}{2}) = (n + \frac{1}{2}).(n - \frac{1}{2})\ldots\frac{3}{2}.\frac{1}{2}\,\Gamma(\frac{1}{2})$$

$$= \frac{(2n+1)(2n-1)\ldots 3.1}{2^{n+1}}\sqrt{\pi},$$

putting $z = \frac{1}{2}$ in the result on p.197, and

$$n! = n.(n-1)\ldots.1$$

$$= \frac{2n(2n-2)\ldots 2.1}{2^n}$$

and hence

$$n!\Gamma(n + \frac{3}{2}) = \frac{(2n+1)!}{2^{n+1}}\sqrt{\pi}.$$

Using this result in the sum above we obtain

$$(\frac{z}{2})^{1/2} J_{1/2}(z) = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = \frac{\sin z}{\sqrt{\pi}},$$

and therefore

$$J_{1/2}(z) = (\frac{2}{\pi z})^{1/2} \sin z.$$

The result for $J_{-1/2}(z)$ is proved similarly.

Since $\frac{1}{2}$ is not an integer the solution of the given differential equation is

$$c_1J_{1/2}(z) + c_2J_{-1/2}(z) = (\frac{2}{\pi})^{1/2}\frac{c_1 \sin z + c_2 \cos z}{\sqrt{z}}$$

$$= \frac{A \sin z + B \cos z}{\sqrt{z}}.$$

14. Taking $\nu = \frac{1}{2}$ in the identity

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z)$$

on p.315 we get

$$J_{3/2}(z) = \frac{1}{z} J_{1/2}(z) - J_{-1/2}(z)$$

$$= (\frac{2}{\pi z})^{1/2}[\frac{\sin z}{z} - \cos z].$$

$J_{-3/2}(z)$ is dealt with similarly.

15. We have that

$$J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n(z/2)^{2n}}{(n!)^2} = 1 - (\frac{z}{2})^2 + \ldots\ldots$$

and so

$$Y_0(z) = \frac{2}{\pi} J_0(z) \int^z \frac{1}{\varsigma\,[J_0(\varsigma)]^2}\, d\varsigma ,$$

as on p.313, has a term $\frac{2}{\pi} J_0(z).\log z$ and so the result follows.

16. Note that the equation should read

$$(xy')' - \frac{n^2}{x} y + \lambda xy = 0$$

or

$$x^2y'' + xy' + (\lambda x^2 - n^2)y = 0.$$

Put $\sqrt{\lambda} x = \xi$, $y(x) = y(\frac{\xi}{\sqrt{\lambda}}) = Y(\xi)$ and then

$$y'(x) = \sqrt{\lambda}\, Y'(\xi),\ y''(x) = \lambda Y''(\xi).$$

Consequently

$$\frac{\xi^2}{\lambda}.\lambda Y'' + \frac{\xi}{\sqrt{\lambda}}.\sqrt{\lambda}Y' + (\lambda.\frac{\xi^2}{\lambda} - n^2)Y = 0,$$

i.e.

$$\xi^2Y'' + \xi Y' + (\xi^2 - n^2)Y = 0.$$

This is Bessel's equation of order n and from the discussion on p.313 the only solution bounded near the origin is a multiple of

$$J_n(\xi) = J_n(\sqrt{\lambda} x) \text{ when } n \text{ is an integer.}$$

For the final part of the problem put

$$y_i(x) = J_n(\sqrt{\lambda_i}\, x),\ y_j(x) = J_n(\sqrt{\lambda_j}\, x)$$

and consider

$$(xy_i')' - \frac{n^2}{x} y_i + \lambda_i xy_i = 0,$$

$$(xy_j')' - \frac{n^2}{x} y_j + \lambda_j xy_j = 0.$$

Multiply the first equation by y_j and the second by y_i and subtract to obtain

$$(x\, y_i')'y_j - (x\, y_j')'y_i = (\lambda_j - \lambda_i)xy_iy_j.$$

Now, integrating by parts,

$$\int_0^b (x\, y_i')'y_j = x\, y_i'\, y_j \Big|_0^b - \int_0^b x\, y_i'\, y_j'$$

$$= -\int_0^b xy_i'\, y_j',$$

since the integrated term is 0 at x = 0 because of the factor x and is 0 at x = b since $y_j(b) = J_n(\sqrt{\lambda_j}\, b) = 0$. We have the same result with i, j interchanged and so

$$(\lambda_j - \lambda_i)\int_0^b x\, y_i\, y_j = 0$$

and if $\lambda_i \neq \lambda_j$ the desired result follows.

17. Put $t = i\theta$ and we get

$$\frac{1}{2\pi i}\int_{-i\pi}^{i\pi} e^{z \sinh t - nt}\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{z \sinh i\theta - in\theta}\, d\theta .$$

Since

$$\sinh i\theta = \frac{e^{i\theta} - e^{-i\theta}}{2} = i \sin \theta$$

we obtain

$$\frac{1}{2\pi i}\int_{-i\pi}^{i\pi} e^{z \sinh t - nt}\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(z \sin\theta - n\theta)} d\theta$$

$$= J_n(z),$$

as shown on p.316.

18. Consider

$$(1 - x^2)P_n'' - 2x\ P_n' + n(n + 1)P_n = 0,$$

$$(1 - x^2)P_m'' - 2x\ P_m' + m(m + 1)P_m = 0.$$

Multiply the first by P_m and the second by P_n and subtract to obtain

$$(n(n + 1) - m(m + 1) \int_{-1}^{1} P_n\ P_m = \int_{-1}^{1} (1 - x^2)(P_m''\ P_n - P_m\ P_n'')$$

$$+ 2 \int_{-1}^{1} x(P_n'\ P_m - P_m'\ P_n).$$

Now

$$\int_{-1}^{1} (1 - x^2)P_m''\ P_n = (1 - x^2)P_m'\ P_n \Big|_{-1}^{1} - \int_{-1}^{1} (- 2x\ P_n + (1 - x^2)P_n')P_m'$$

$$= 2 \int_{-1}^{1} x\ P_n\ P_m' - \int_{-1}^{1} (1 - x^2)P_n'\ P_m',$$

with a similar result with m, n interchanged. Hence

$$(n(n + 1) - m(m + 1))\int_{-1}^{1} P_n\ P_m = 0$$

and when $m \neq n$ the result follows.

19. Consider Rodrigues' formula

$$P_n(x) = \frac{1}{2^n n!}\ \frac{d^n}{dx^n}[(x^2 - 1)^n].$$

Then

$$\int_{-1}^{1} P_n{}^2(x)dx = (\frac{1}{2^n n!})^2 \int_{-1}^{1} \frac{d^n}{dx^n}[(x^2 - 1)^n].\frac{d^n}{dx^n}[(x^2 - 1)^n]dx$$

If we integrate by parts n times the integrated term will be 0 at $x = \pm 1$ at each stage because of a factor $x^2 - 1$ and each integration will introduce a factor -1. Hence

$$\int_{-1}^{1} P_n{}^2(x)dx = \frac{(-1)^n}{(2^n n!)^2} \int_{-1}^{1} (x^2 - 1)^n.\frac{d^{2n}}{dx^{2n}}[(x^2 - 1)^n]dx$$

and

$$\frac{d^{2n}}{dx^{2n}}[(x^2 - 1)^n] = \frac{d^{2n}}{dx^{2n}}[x^{2n} + \text{lower powers of } x] = (2n)!$$

and so, using $(x^2 - 1)^n = (-1)^n(1 - x^2)^n$,

$$\int_{-1}^{1} P_n^2(x)dx = \frac{(2n)!}{(2^n n!)^2} \int_{-1}^{1} (1 - x^2)^n dx$$

$$= \frac{2.(2n)!}{(2^n n!)^2} \int_{0}^{1} (1 - x^2)^n dx,$$

since the integrand is an even function. Put $x = \cos\theta$ and then

$$\int_0^1 (1 - x^2)^n dx = \int_0^{\pi/2} \sin^{2n+1}\theta \, d\theta .$$

We have

$$\int_0^{\pi/2} \sin^{2k+1}\theta \, d\theta = \int_0^{\pi/2} \sin^{2k-1}\theta \, d\theta - \int_0^{\pi/2} \sin^{2k-1}\theta \cos^2\theta \, d\theta$$

and

$$\int_0^{\pi/2} \sin^{2k-1}\theta.\cos\theta.\cos\theta \, d\theta = \frac{\sin^{2k}\theta}{2k}.\cos\theta \Big|_0^{\pi/2} + \int_0^{\pi/2} \frac{\sin^{2k+1}\theta}{2k} d\theta$$

$$= \int_0^{\pi/2} \frac{\sin^{2k+1}\theta}{2k} d\theta .$$

Therefore

$$\int_0^{\pi/2} \sin^{2k+1}\theta \, d\theta = \frac{2k}{2k + 1} \int_0^{\pi/2} \sin^{2k-1}\theta \, d\theta .$$

Repeated applications of this result gives

$$\int_0^{\pi/2} \sin^{2n+1}\theta \, d\theta = \frac{2n}{2n + 1}.\frac{2n - 2}{2n - 1} \cdots\cdots \frac{2}{3}.\int_0^{\pi/2} \sin\theta \, d\theta$$

$$= \frac{[2n.(2n - 2) \ldots 2]^2}{(2n + 1).2n(2n - 1) \ldots 3.2}$$

$$= \frac{(2^n n!)^2}{(2n + 1)!}.$$

From the above

$$\int_{-1}^{1} P_n^2(x)dx = \frac{2.(2n)!}{(2^n n!)^2}.\frac{(2^n n!)^2}{(2n + 1)!}$$

$$= \frac{2}{2n + 1}.$$

20. We use Rodrigues' formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}[(x^2 - 1)^n].$$

The polynomial $(x^2 - 1)^n$ has n zeros at $x = -1$ and n zeros at $x = 1$. By Rolle's theorem $\frac{d}{dx}[(x^2 - 1)^n]$ has $n - 1$ zeros at $x = -1$, $n - 1$ zeros at $x = 1$ and a zero $x_1^{(1)}$ satisfying $-1 < x_1^{(1)} < 1$. Again applying Rolle's theorem, $\frac{d^2}{dx^2}[(x^2 - 1)^n]$ has $n - 2$ zeros at $x = -1$, $n - 2$ zeros at $x = 1$ and a zero $x_1^{(2)}$ satisfying $-1 < x_1^{(2)} < x_1^{(1)}$ and a zero $x_2^{(2)}$ satisfying $x_1^{(1)} < x_2^{(2)} < 1$. We continue this argument for n steps and note that at each stage the zeros in the open interval $(-1, 1)$ are distinct. At the nth stage we see that

$$\frac{d^n}{dx^n}[(x^2 - 1)^n] \text{ has n distinct, i.e. simple, zeros in } (-1, 1).$$

21. If we start with the associated Legendre equation instead of the Legendre equation as we did in the solution to p.18, then the argument proceeds in precisely the same way as in the earlier solution.

22. We argue as in the solution to P.19. On p.323,

$$P_n^m(x) = (-1)^n(1 - x^2)^{m/2} \frac{d^m}{dx^m} P_n(x)$$

$$= (-1)^n(1 - x^2)^{m/2} . \frac{1}{2^n n!} \frac{d^{m+n}}{dx^{m+n}}[(x^2 - 1)^n].$$

If $m > n$, then $m + n$ exceeds $2n$, the degree of $(x^2 - 1)^n$, and so the above derivative is 0. It is usual, cf. the Γ-function, to put $(-k)! = \infty$, $\frac{1}{(-k)!} = 0$ for $k = 1, 2, \ldots$. Hence the result of the problem is true for $m > n$ and so we assume that $m \leq n$.

Consider

$$\int_{-1}^{1} [P_n^m(x)]^2 dx$$

$$= \left(\frac{1}{2^n n!}\right)^2 \int_{-1}^{1} \{(1 - x^2)^m \frac{d^{m+n}}{dx^{m+n}}[(x^2 - 1)^n]\} . \frac{d^{m+n}}{dx^{m+n}}[(x^2 - 1)^n] dx.$$

We integrate by parts $m + n$ times with the differentiations applying to the factor that arises form the function in brace brackets. This function is a polynomial of degree

$$2n - (m + n) + 2m = m + n$$

with highest coefficient

$$(-1)^m \, 2n(2n - 1) \ldots (2n - m - n + 1)$$

and so after m + n differentiations we arrive at the constant

$$(m + n)!.(-1)^m \, 2n(2n - 1) \ldots (n - m + 1)$$

$$= (-1)^m.(2n)! \, \frac{(n + m)!}{(n - m)!}$$

At each integration by parts the integrated term is 0 at x = 1, x = -1 because of a factor $x^2 - 1$ and a factor (-1) is introduced. Hence

$$\int_{-1}^{1} [P_n^m(x)]^2 dx = (\frac{1}{2^n n!})^2 (-1)^m (2n)! \, \frac{(n + m)!}{(n - m)!} (-1)^{m+n} \int_{-1}^{1} (x^2 - 1)^n dx$$

$$= \frac{(2n)!}{(2^n n!)^2} \, \frac{(n + m)!}{(n - m)!} \int_{-1}^{1} (1 - x^2)^n dx$$

$$= \frac{2}{2n + 1} \, \frac{(n + m)!}{(n - m)!},$$

using the value of $\int_{-1}^{1} (1 - x^2)^n dx$ found in the solution to P.19.

23. The left hand side of the given equation is of the general form on p.325 with

$$p(x) = e^{-3x}, \; q(x) \equiv 0, \; \rho(x) = e^{-3x}$$

Considering the transformation at the top of p.326, where

$t = \int^x (\frac{\rho}{p})^{1/2} dx$, we can take t = x and $u(t) = e^{-3/2x} \, y(x)$. Then

$$\dot{u}(t) = -\frac{3}{2} e^{-3/2x} \, y(x) + e^{-3/2x} \, y'(x),$$

$$\ddot{u}(t) = \frac{9}{4} e^{-3/2} \, y(x) - 3^{-3/2x} \, y'(x) + e^{-3/2x} \, y''(x)$$

$$= \frac{9}{4} u(t) + e^{3/2x} (e^{-3x} \, y'(x))'$$

$$= \frac{9}{4} u(t) + e^{3/2x} [f(x) - \lambda e^{-3/2x} \, u(t)]$$

and so

$$\ddot{u} + (\lambda - \frac{9}{4}) u = e^{3/2t} \, f(t).$$

Therefore $q(t) \equiv -\frac{9}{4}$, $q(t) = e^{3/2t} \, f(t)$.

24. Here we are considering a = 0, b = π and $\alpha = \beta = 0$ in the context of (a), (b) on p.326. The homogeneous equation $y'' + \lambda y = 0$ has the general solution

$$y(x) = A \cos \lambda x + B \sin \lambda x, \quad y'(x) = -\lambda A \sin \lambda x + \lambda B \cos \lambda x$$

and hence

$$\phi(x, \lambda) = \frac{\sin \lambda x}{\lambda}, \quad \psi(x, \lambda) = \frac{1}{\lambda} \sin \{\lambda(\pi - x)\}.$$

ϕ, λ are obtained by choosing A, B so that $y(0) = 0$, $y'(0) = -1$ and $y(\pi) = 0$, $y'(\pi) = -1$ respectively as on p.327. In this case

$$\begin{aligned} W(\phi, \psi) &= \phi\psi' - \phi'\psi \\ &= -\frac{\sin \lambda x}{\lambda}.-\cos \lambda(\pi - x) - [-\cos \lambda x.\frac{\sin \lambda(\pi - x)}{\lambda}] \\ &= \frac{1}{\lambda}[\sin \lambda x \cos \lambda(\pi - x) + \cos \lambda x.\sin \lambda(\pi - x)] \\ &= \frac{\sin \lambda\pi}{\lambda} = W(\lambda). \end{aligned}$$

Hence the eigenvalues are ±1, ±2, ...

If $\lambda = k$, a non-zero integer, then

$$\phi(x, k) = -\frac{\sin kx}{k}, \quad \psi(x, k) = \frac{1}{k} \sin k(\pi - x) = \frac{(-1)^{k-1}}{k} \sin kx$$

and so, in the notation of p.332, $A_k = (-1)^{k-1}$. Also

$$w'(k) = -\cos k\pi = (-1)^{k-1} \text{ and consequently, as on p.333,}$$

$$\begin{aligned} \Phi_k(x) &= [\frac{A_k}{w'(k)}]^{1/2} \phi(x, k) \\ &= -\frac{\sin kx}{k}. \end{aligned}$$

From this the form of the expansion result in this particular case follows from Theorem 7.7.4.

25. One can check that $\lambda = 0$ is not an eigenvalue of the Sturm-Liouville problem and so the Green's function is determined by properties 1-4. on p.340. In this case $p(x) \equiv 1$, $q(x) \equiv 0$ and so 1. gives, for each x in (0, 1),

$$G(t) = G(x, t) = \begin{cases} A_1 t + B_1 & (0 < t < x) \\ A_2 t + B_2 & (0 < t < 1), \end{cases}$$

where $A_1, \ldots, B_2$ are independent of t. Property 2. shows that

$$A_1 x + B_1 = A_2 x + B_2;$$

and property 3. shows that

$$B_1 = 0, \quad A_2 + B_2 = 0$$

since in this case $a = 0$, $b = 1$ and $\alpha = \beta = 0$. Finally property 4. shows that

$$A_2 - A_1 = -1.$$

Solving for $A_1, \ldots, B_2$ we obtain

$$G(x, t) = \begin{cases} (1 - x)t & (0 < t < x) \\ x(1 - t) & (x < t < 1). \end{cases}$$

For this problem $\rho(x) \equiv 1$ and so, as on p.342,

$$K(x, t) = \sqrt{\rho(x)\rho(t)}\ G(x, t) = G(x, t),$$

$$u(x) = \sqrt{\rho(x)}.y(x) = y(x),$$

$$F(x) = \int_0^1 [\sqrt{\rho(x)\rho(t)}\ G(x, t).f(t)/\sqrt{\rho(t)}]dt$$

$$= \int_0^1 G(x, t)f(t)dt,$$

and hence

$$U(x) = F(x) + \lambda \int_0^1 K(x, t)u(t)dt$$

is the associated Fredholm integral equation.

The kernel K is determined from the homogeneous equation and F is determined from f and the homogeneous equation. The solution $y(x)$ of the differential equation with boundary conditions and the solution of the integral equation are related by $u(x) = \sqrt{\rho(x)}\ y(x) = y(x)$ so the solution of the problem in one form gives immediately the solution in the other form.

26. One can argue directly as in the earlier problem or first obtain an equivalent problem as follows. Put $x = e^{\xi}$, $\xi = \log x$ and then, with $y(e^{\xi}) = Y(\xi)$,

$$\frac{d}{dx} y(x) = Y'(\xi)\frac{d\xi}{dx}, \quad xy'(x) = Y'(\xi);$$

$$\frac{d}{dx}[xy'(x)] = Y''(\xi)\frac{d\xi}{dx} = e^{-\xi}\ Y''(\xi).$$

Hence the equation of the problem becomes

$$e^{-\xi}\ Y''(\xi) + \lambda e^{\xi} Y(\xi) = -f(e^{\xi}) \qquad (0 \le \xi \le 1)$$

or

$$Y''(\xi) + \lambda e^{2\xi} Y(\xi) = -e^{\xi} f(e^{\xi}) \qquad (0 \le \xi \le 1),$$

with boundary conditions $Y(0) = Y(1) = 0$. The Green's function in this case is the same as in the previous problem, but now $\rho = \rho(\xi) = e^{2\xi}$ and f is replaced by $e^{\xi} f(e^{\xi})$. One can now find the Fredholm equation in this case for $U(\xi)$, as on p.342, and setting $U(\log x) = u(x)$ and changing the variable of integration from t on [0, 1] to τ on [1, e] by the transformation $\tau = e^{t}$ we find the Fredholm equation for u.

Chapter 8

Fourier Transforms

| READ §8.1 |

This section is rather well presented and deals with a number of the basic results for Fourier series.

Self-Assessment Questions

SAQ 1, 2, 3 are Exercises 8.1, #1, 2, 3.

| READ §8.2 |

Theorem 8.2.1 The 'Cauchy principal value' means that we consider $\lim_{R\to\infty} \frac{1}{2\pi}\int_{-R}^{R}\int_{-\infty}^{\infty}$ as in the fourth line of the proof. For the integral to exist in the ordinary sense we would require $\lim_{R_1,R_2\to\infty} \frac{1}{2\pi}\int_{-R_1}^{R_2}\int_{-\infty}^{\infty}$, where we consider R_1, $R_2 \to \infty$ independently.

For example, $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ exists in the latter sense since

$$\int_{-R_1}^{R_2} \frac{dx}{1+x^2} = \tan^{-1}x \Big|_{-R_1}^{R_2} = \underbrace{\tan^{-1}R_2 + \tan^{-1}R_1}_{\to\,\pi}$$

as R_1, $R_2 \to \infty$ in any manner, i.e. given $\varepsilon > 0$ there is an $R_0(\varepsilon)$ such that for all R_1, $R_2 > R_0(\varepsilon)$,

$$\left|\int_{-R_1}^{R_2} \frac{dx}{1+x^2} - \pi\right| < \varepsilon .$$

On the other hand for R_1, $R_2 > 0$,

$$\int_{-R_1}^{R_2} \frac{xdx}{1+x^2} = \int_0^{R_2} \frac{xdx}{1+x^2} - \int_0^{R_2} \frac{xdx}{1+x^2}$$

$$= \frac{1}{2} \log\left(\frac{1+R_2^2}{1+R_1^2}\right)$$

and so $\int_{-\infty}^{\infty} \frac{xdx}{1+x^2}$ does not exist in the usual sense. However, if $R_1 = R_2$, then $\int_{-R}^{R} \frac{xdx}{1+x^2} = 0$ so that the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{xdx}{1+x^2}$ is 0.

Self-Assessment Questions

SAQ 4. If $\int_{-\infty}^{\infty} |f(t)|\,dt$ exists prove that $G(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau)e^{-ix\tau}d\tau$ is continuous on $(-\infty, \infty)$ and $G(x) \to 0$ as $x \to \pm\infty$.

SAQ 5. Exercises 8.2, #1.

| READ §8.3 |

Theorem 8.3.4 This result should be compared with Theorem 8.1.3.

Problems for §8.1, §8.2, §8.3

P.1, 2 are Exercises 8.1, #6, 7.

P.3, 4 are Exercises 8.2, #2, 3

P.5, 6, 7, 8, 9 are Exercises 8.3, #1, 2, 4, 5, 6.

[Note: Exercises 8.3, #5 is badly stated. Part of P.8 is finding the correct statement of this exercise.]

| READ §8.4 |

Theorem 8.4.6 At the start of the proof there is differentiation under the integral sign. When the limits of integration are ∞ and −∞ justification of this can be quite tricky. Generally speaking if this operation leads to a convergent integral, then one expects this operation to be valid.

p.372 The argument used to show that $e^{\gamma t}h(t)$ is piecewise continuous is rather unsatisfactory. It is best to assume that those f, g to be considered will always lead to an h with sufficiently nice properties for the application that is being considered.

Problems for § 8.4

P.10, 11, 12, 13, 14, 15 are Exercises 8.4, #1, 2, 3, 4, 5, 6.

| READ § 8.5 |

Theorem 8.5.1 In this theorem we consider

$$y = F^{-1}[F[y]] = F^{-1}[\frac{F[f]}{P(iz)}]$$

in a somewhat formal manner under hypotheses that ensure that the integral for the inverse on the right exists. But instead of proving this integral is the inverse, the result is verified by assuming differentiation under the integral sign is permissible. A proper justification for such an argument can be rather tricky in a general context and it is perhaps better to check that everything is all right in each particular case.

p.379 The discussion on the one-sided Fourier transform is to be superceded by the Laplace transform in the next chapter and so one should not spend too much time on the details.

Problems for § 8.5

P.16, 17, 18 are Exercise 8.5, #1, 2, 3.

| READ § 8.6 |

The examples in this section illustrate the way in which the theory of Fourier transforms can be applied to partial differential equations. Note that the emphasis is on Fourier transforms and not on the 'easiest' or 'best' way of solving a particular problem. For the wave equation, for example, d'Alembert's solution should always be kept in mind, i.e.

$u(x, t) = f(x + at) + f(x - at)$ is a solution of $\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}$.

The solutions of the examples could be better presented in some cases and various implicit assumptions are made. It is better, I think, to regard the arguments given as routines for arriving at possible answers which require to be checked directly. However, in some cases such verification can be rather difficult.

As an illustration of the points made above consider Example 8.6.1. To obtain

$$(iz)^2 U(z, t) = \frac{1}{a^2} \frac{\partial^2 U}{\partial t^2}$$

we consider

$$\int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{-izx} dx = \frac{1}{a^2} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial t^2} e^{-izx} dx.$$

The integral on the left is integrated twice by parts to give

$$\int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{-izx} dx = \frac{\partial u}{\partial x} e^{-izx} \Big|_{-\infty}^{\infty} + iz \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} e^{-izx} dx$$

$$= \frac{\partial u}{\partial x} e^{-izx} \Big|_{-\infty}^{\infty} + iz.ue^{-izx} \Big|_{-\infty}^{\infty} + (iz)^2 \int_{-\infty}^{\infty} ue^{-izx} dx.$$

The given condition on u(x, t) as $|x| \to \infty$ ensures that the second term on the right is 0 when z is real, but not the first term. For this we require an additional condition, viz. $\frac{\partial u}{\partial x} \to 0 \ (|x| \to \infty)$.

To obtain the equation for U we must assume that

$$\int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial t^2} e^{-izx} dx = \frac{\partial^2}{\partial t^2} \int_{-\infty}^{\infty} ue^{-izx} dx$$

$$= \frac{\partial^2 U}{\partial \tau^2},$$

i.e. that the operations of integration from $-\infty$ to ∞ and partial differentiation of the 2nd order can be interchanged. This would require quite severe a priori restrictions on the 'unknown' function u(x, t) if one wished to be rigorous.

Example 8.6.1 illustrates some of the main features of the use of Fourier transforms in §8.6. One has two variables x, t and one takes the Fourier transform of u(x, t) with respect to one of them. After integration by parts and an interchange of differentiation and integration one obtains an ordinary differential equation. In each integration by parts the integrated part must be 0 at both limits. In order to ensure this, the transformation variable must be suitably chosen and an appropriate form of the transform used.

The ordinary differential equation is solved taking account of the initial and boundary conditions and the solution of the original problem is then found by applying the inverse Fourier transform. It may not be clear to begin with which of x and t should be chosen as the transformation variable. If, having chosen one of x and t, one runs into an impasse then one must start again with the other variable.

It may happen, for example, that for $t > 0$ one can check that a 'solution' one has obtained is correct, but at $t = 0$ the solution has some kind of peculiar singularity. In some way this may reflect a peculiarity of the physical system. When one shakes a string or applies heat the physical phenomena at the 'interface' are probably extremely complex and perhaps a strange singularity of a solution is the mathematical counterpart of this complexity.

Problems for §8.6

P.19, 20, 21 are Exercises 8.6, #1, 2, 3.

§8.7 is not part of M828.

Solutions to Self-Assessment Questions

1. Define $F(t) = f(\frac{L}{\pi}t)$ so that F on$(-\pi, \pi)$ corresponds to f on $(-L, L)$, i.e. $F(\pi) = f(L)$, $F(-\pi) = f(-L)$. Applying Theorem 8.1.5 we obtain

$$\frac{F(\tau^+) + F(\tau^-)}{2} = \frac{A_0}{2} + \sum_{k=1}^{\infty} (A_k \cos k\tau + B_k \sin k\tau),$$

where

$$A_k = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\cos kt\,dt, \quad B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\sin kt\,dt.$$

Put $u = \frac{Lt}{\pi}$ and then

$$A_k = \frac{1}{\pi}\int_{-L}^{L} f(u)\cos\frac{k\pi u}{L}\cdot\frac{\pi}{L}\,du = \frac{1}{L}\int_{-L}^{L} f(u)\cos\frac{k\pi u}{L}\,du = a_k,$$

with a similar result showing that $B_k = b_k$. Finally put $t = \frac{\pi}{L}\tau$ in the series above to obtain the desired result.

2. In the series of the preceding problem put

$$\cos \frac{k\pi}{L} t = \frac{e^{ik\pi/Lt} + e^{-ik\pi/Lt}}{2},$$

$$\sin \frac{k\pi}{L} t = \frac{e^{ik\pi/Lt} - e^{-ik\pi/Lt}}{2i},$$

to obtain the series of this problem with

$$C_k = \frac{a_k}{2} + \frac{b_k}{2i} = \frac{1}{2}(a_k - ib_k),$$

$$C_{-k} = \frac{a_k}{2} - \frac{b_k}{2i} = \frac{1}{2}(a_k + ib_k).$$

3. We have the result of the first problem with

$$a_k = \frac{1}{L}\int_{-L}^{L} f(t) \cos \frac{k\pi}{L} t \, dt$$

$$= \frac{1}{L}\left(\int_{0}^{L} + \int_{-L}^{0}\right) f(t) \cos \frac{k\pi}{L} t \, dt$$

and if $t = -\tau$,

$$\int_{-L}^{0} f(t) \cos \frac{k\pi}{L} t \, dt = \int_{L}^{0} f(-\tau) \cos \left(\frac{k\pi}{L}.-\tau\right) (-d\tau)$$

$$= - \int_{0}^{L} f(\tau) \cos \frac{k\pi}{L}.\tau \, d\tau .$$

Hence $a_k = 0$ and similarly

$$\frac{1}{L}\int_{-L}^{L} f(t) \sin \frac{k\pi}{L} t \, dt = \frac{1}{L}\left(\int_{0}^{L} + \int_{-L}^{0}\right) f(t) \sin \frac{k\pi}{L} t \, dt$$

$$= \frac{1}{L}\left(\int_{0}^{L} + \int_{0}^{L}\right) f(t) \sin \frac{k\pi}{L} t \, dt.$$

4. Given $\varepsilon > 0$ choose $X = X(\varepsilon) > 0$ so that

$$\int_{-\infty}^{-X} |f(t)|dt < \varepsilon, \int_{X}^{\infty} |f(t)|dt < \varepsilon .$$

Then, considering $\int_{-\infty}^{\infty} = \int_{-\infty}^{-X} + \int_{-X}^{X} + \int_{X}^{\infty}$,

$$|G(x + h) - G(x)| < \frac{4\varepsilon}{\sqrt{2\pi}} + \frac{1}{\sqrt{2\pi}}\left|\int_{-X}^{X} f(\tau)[e^{-i(x+h)\tau} - e^{-ix\tau}]d\tau\right|$$

$$\leq \frac{4\varepsilon}{\sqrt{2\pi}} + \frac{K}{\sqrt{2\pi}} \max_{-X\leq\tau\leq X} |e^{-ih\tau} - 1|,$$

where $K = \int_{-\infty}^{\infty} |f(t)|dt$. Now e^{-it} is continuous in t and is 1 when $t = 0$ and so there is a $t_0 = t_0(\varepsilon) > 0$ such that

$$|e^{-it} - 1| < \varepsilon \quad (|t| < t_0)$$

and hence

$$|e^{-ih\tau} - 1| < \varepsilon \quad (-X \leq \tau \leq X, \ |h| < \frac{t_0}{X} = h_0, \text{ say}).$$

Consequently,

$$|G(x + h) - G(x)| < \frac{\varepsilon}{\sqrt{2\pi}} (4 + K)$$

for $|h| < h_0$, i.e. G is uniformly continuous on $(-\infty, \infty)$.

From Theorem 8.1.1,

$$\int_{-X}^{X} f(\tau)\, e^{-ix\tau} d\tau = \int_{-X}^{X} f(\tau) \cos x\tau \, d\tau + i \int_{-X}^{X} f(\tau) \sin x\tau \, d\tau$$

$$\to 0 \quad (x \to \infty)$$

and so

$$\left|\int_{-X}^{X} f(\tau)\, e^{-ix\tau} d\tau\right| < \varepsilon \quad (|x| > x_0),$$

where $x_0 = x_0(\varepsilon) > 0$. Hence

$$\left|\int_{-\infty}^{\infty} f(\tau)\, e^{-ix\tau} d\tau\right| < 3\varepsilon \, (|x| > x_0),$$

i.e. $G(x) \to 0$ $(x \to \pm\infty)$.

Remark If $a < b$ are finite and f is Riemann integrable over [a, b] then given $\varepsilon > 0$ there is a function g continuous on [a, b] such that

$$\left|\int_{a}^{b} f(t)\, dt - \int_{a}^{b} g(t)\, dt\right| < \varepsilon .$$

This result shows that Theorem 8.1.1 is true for any integrable function.

5. The Fourier transform is

$$G(x) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-a\tau} . e^{-ix\tau} \, d\tau$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-(a+ix)\tau}\, d\tau$$

$$= \frac{1}{\sqrt{2\pi}} . -\frac{1}{a+ix} . e^{-(a+ix)\tau} \Big|_0^\infty$$

$$= \frac{1}{\sqrt{2\pi}} . \frac{1}{a+ix} .$$

For $t > 0$ consider the integral of $G(z)e^{izt}$ round the contour C of Figure 8.2.1 on p.363. As $R \to \infty$, the integral over the semi-circle tends to 0 and therefore

$$\lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} G(x)e^{ixt} dx = \frac{2\pi i}{\sqrt{2\pi}} . R,$$

where R is the sum of the residues of $\frac{e^{izt}}{a+iz}$ in the upper half-plane. This function has a simple role at $z = ai$ and so $R = \frac{1}{i}e^{-at}$ and

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(x)e^{ixt}\, dx = e^{-at} .$$

For $t < 0$ we consider the semi-circle from R to -R in the lower half plane and for $t = 0$ we evaluate $\int_{-R}^{R} G(x)dx$ directly and then let $R \to \infty$.

Solutions to Problems

1. If $f'(t)$ has the Fourier series

$$\frac{A_0}{2} + \sum_{k=1}^{\infty} (A_k \cos kt + B_k \sin kt)$$

then, if t_0 is a point of continuity of $f'(t)$,

$$f'(t_0) = \frac{A_0}{2} + \sum_{k=1}^{\infty} (A_k \cos kt_0 + B_k \sin kt_0),$$

by Theorem 8.1.5. Now

$$A_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(t) \cos kt\, dt$$

$$= \frac{1}{\pi} f(t) \cos kt \Big|_{-\pi}^{\pi} + \frac{k}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt\, dt$$

$$= k . b_k ,$$

where b_k is the kth Fourier sine coefficient of $f(t)$. The integrated term is 0 since $f(\pi) = f(-\pi)$ by the periodicity of $f(t)$. Similarly

$$B_k = -ka_k,$$

where a_k is the kth Fourier cosine coefficient of $f(t)$. Also

$$A_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(t)\, dt$$

$$= \frac{1}{\pi} [f(\pi) - f(-\pi)]$$

$$= 0.$$

Hence

$$f'(t) = \sum_{1}^{\infty} [k\, b_k \cos kt - k\, a_k \sin kt]$$

and this series is the Fourier series of $f(t)$ differentiated term by term.

2. If a periodic function is differentiable so is its derivative since differentiating $f(x + 2\pi) = f(x)$ gives $f'(x + 2\pi) = f'(x)$. Now

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt\, dt$$

$$= \frac{1}{\pi} \frac{\sin kt}{k} f(t) \Big|_{-\pi}^{\pi} - \frac{1}{k\pi} \int_{-\pi}^{\pi} f'(t) \sin kt\, dt$$

$$= \frac{1}{k\pi} \cdot \int_{-\pi}^{\pi} f'(t) \sin kt\, dt.$$

Repeating this procedure n times we find that

$$a_k = \pm \frac{1}{k^n \pi} \int_{-\pi}^{\pi} f^{(n)}(t)\, [?]\, dt,$$

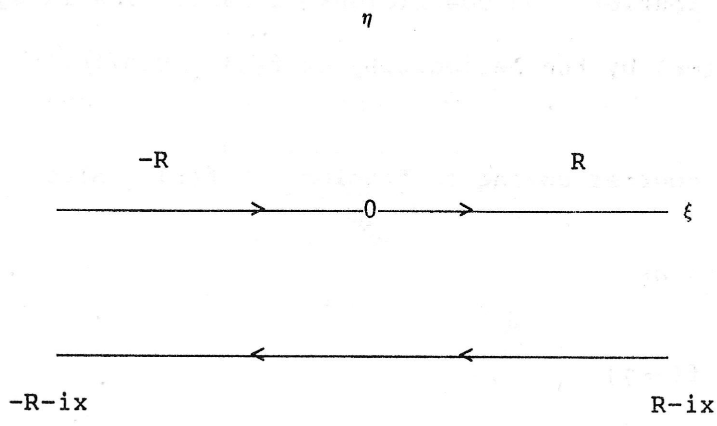
where [?] is either sin kt or cos kt. From Theorem 8.1.1 it follows that $k^n a_k \to 0$ $(k \to \infty)$ and $k^n b_k$ is dealt with in the same way.

3. Note that

$$F[e^{-t^2/2}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} . e^{-ixt} dt$$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(t+ix)^2}\, dt.$$

Suppose, for convenience, that $x > 0$ and consider $\int e^{-1/2(\zeta+ix)^2} d\zeta$ round the pictured contour with $\zeta = \xi + i\eta$.



The integrand is an entire function of ζ and so the integral is 0. The integrals over the segments [R, R - ix] and [-R - ix, -R] tend to 0 as R → ∞ and so

$$\int_{-\infty}^{\infty} e^{-1/2(t+ix)^2}\,dt = \int_{-\infty-ix}^{\infty-ix} e^{-1/2(\zeta+ix)^2}\,d\zeta$$
$$= \int_{-\infty}^{\infty} e^{-\xi^2/2}\,d\xi$$
$$= \sqrt{2\pi}\ .$$

For the value of the final integral note that

$$\sqrt{\pi} = \Gamma(\tfrac{1}{2}) = \int_0^{\infty} \frac{e^{-t}}{\sqrt{t}}\,dt$$
$$= \int_0^{\infty} (\tfrac{\xi}{\sqrt{2}})^{-1}\, e^{-\xi^2/2}.\xi\ d\xi \qquad (t = \xi^2/2)$$
$$= \sqrt{2}\int_0^{\infty} e^{-\xi^2/2}\,d\xi$$

and so

$$\int_{-\infty}^{\infty} e^{-\xi^2/2}\,d\xi = 2\int_0^{\infty} e^{-\xi^2/2}\,d\xi = 2.\sqrt{\tfrac{\pi}{2}}.$$

Hence result for $F[e^{-t^2/2}]$.

4. If f is real-valued then we can replace the integrand in Theorem 8.2.1 by its real part and

$$\mathrm{Re}[f(\tau)e^{ix(t-\tau)}] = f(\tau)\ \mathrm{Re}[e^{ix(t-\tau)}]$$
$$= f(\tau)\ \cos[x(t-\tau)].$$

Note that

$$f(\tau)\cos[x(t-\tau)] = f(\tau)[\cos xt.\cos x\tau + \sin xt.\sin x\tau]$$

and if $f(\tau)$ is an even function then for x, t fixed $f(\tau).\sin xt.\sin x\tau$ is an odd function. Hence the integral of this function with respect to τ over $(-\infty, \infty)$ is 0 and this is the second of the results asked for in the notation of Theorem 8.2.1. The last result is proved similiarly.

5. Note that

$$G(z) = F[e^{-|t|}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|\tau|}.e^{-i\tau z}d\tau$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_0^{\infty} e^{-\tau-i\tau z}d\tau + \int_{-\infty}^{0} e^{\tau-i\tau z}d\tau\right].$$

If $z = x + iy$, then

$$\begin{aligned}\mathrm{Re}[-|\tau| - i\tau z] &= -|\tau| + \tau y\\ &= -|\tau|\,[1 - \sigma y],\end{aligned}$$

where

$$\sigma = \begin{array}{l} 0(\tau = 0)\\ 1(\tau > 0)\\ -1(\tau < 0).\end{array}$$

The integral $\int_{-\infty}^{\infty} e^{-|\tau|-i\tau z}d\tau$ will converge at both limits provided

$1 - \sigma y > 0$, i.e. provided $|y| < 1$ and so $G(z)$ is analytic in the strip

$$\{z: z = x + iy, -\infty < x < \infty, -1 < y < 1\}.$$

For such z,

$$\begin{aligned}\int_0^{\infty} e^{-\tau(1+iz)}d\tau &= -\frac{1}{1+iz}.e^{-\tau(1+iz)}\Big|_0^{\infty}\\ &= \frac{1}{1+iz};\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{0} e^{\tau(1-iz)}d\tau &= \frac{1}{1-iz}\,e^{\tau(1-iz)}\Big|_{-\infty}^{0}\\ &= \frac{1}{1-iz}.\end{aligned}$$

For the final part of the problem we must consider for $-1 < y < 1$,

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty+i\gamma}^{\infty+i\gamma} e^{itz}G(z)\,dz &= \frac{1}{\sqrt{2\pi}}\int_{-\infty+i\gamma}^{\infty+i\gamma} e^{itz}.\sqrt{\frac{2}{\pi}}.\frac{1}{1+z^2}\,dz\\ &= \frac{1}{\pi}\int_{-\infty+i\gamma}^{\infty+i\gamma}\frac{e^{itz}}{1+z^2}\,dz.\end{aligned}$$

From Corollary 8.3.1 we take $\gamma = 0$ for convenience.

For $t > 0$, consider $\int \frac{e^{itz}}{1 + z^2} dx$ round the contour of Figure 8.2.1. For $R > 1$ the integrand has a simple pole at $z = i$ with residue

$$\left.\frac{e^{itz}}{2z}\right|_{z=i} = \frac{e^{-t}}{2i}.$$

As $R \to \infty$, the integral along the semi-circle tends to 0 and so, letting $R \to \infty$,

$$\int_{-\infty}^{\infty} \frac{e^{itz}}{1 + z^2} dz = 2\pi i.\frac{e^{-t}}{2i} = \pi e^{-t},$$

and so the inverse in this case is

$$\frac{1}{\pi}.\pi e^{-t} = e^{-t} = e^{-|t|}.$$

For $t < 0$ we proceed as above with the contour having a semi-circle in the lower half-plane.

6. In this case

$$G(z) = F[u(t) \cos \omega t] = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \cos \omega t.e^{-itz} dt$$

$$= \frac{1}{\sqrt{2\pi}}.\int_0^{\infty} \frac{e^{i\omega t} + e^{-i\omega t}}{2}.e^{-itz} dt.$$

As in Example 8.3.1, p.366,

$$G(z) = \frac{1}{2\sqrt{2\pi}} \left[\frac{i}{\omega - z} - \frac{i}{\omega + z}\right]$$

$$= \frac{1}{2\sqrt{2\pi}}.\frac{2iz}{\omega^2 - z^2}$$

$$= \frac{1}{\sqrt{2\pi}}.\frac{iz}{\omega^2 - z^2}.$$

To obtain this result we require

$$e^{i(\omega-z)t} \to 0, \; e^{-i(\omega+z)t} \to 0 \quad (t \to \infty)$$

and so we require $\mathrm{Re}[iz] > 0$, $\mathrm{Im}\, z < 0$.

The inversion of the transform is essentially the same as in Example 8.3.1.

7. We are considering

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty+i\gamma}^{\infty+i\gamma} G(z)\, e^{izt} dz = \frac{1}{2\pi i} \int_{-\infty+i\gamma}^{\infty+i\gamma} \frac{e^{izt}}{z} dz$$

in the two cases: $\gamma > 0$ and $\gamma < 0$. If $\gamma > 0$, then

$$\int_{-\infty+i\gamma}^{\infty+i\gamma} \frac{e^{izt}}{z}\,dz = \int_{-\infty}^{\infty} \frac{e^{ixt}.e^{-\gamma t}}{x + i\gamma}\,dx$$

and for convergence we must assume that $t > 0$. Considering $\int \frac{e^{izt}}{z}\,dz$ round a contour like that in Figure 8.3.1, except that the segment $[-R + i\gamma, R + i\gamma]$ is in the upper half-plane,

$$\frac{1}{2\pi i}\int_{-\infty+i\gamma}^{\infty+i\gamma} \frac{e^{izt}}{z}\,dz = 0.$$

If $t < 0$, then we consider the semi-circle in the lower half-plane. In this case the integrand has a simple pole at $z = 0$ within the contour and hence one shows that in this case

$$\frac{1}{2\pi i}\int_{-\infty+i\gamma}^{\infty+i\gamma} \frac{e^{izt}}{z}\,dz = -1.$$

When $\gamma < 0$ we prove similarly that

$$\frac{1}{2\pi i}\int_{-\infty+i\gamma}^{\infty+i\gamma} \frac{e^{izt}}{z}\,dz = \begin{cases} 1 & (t > 0) \\ 0 & (t < 0) \end{cases}.$$

Interpretation

If $G(z) = \frac{1}{i\sqrt{2\pi}}.\frac{1}{z}$ then it has, from the point of view considered in §8.3, two domains of analyticity, viz. the upper half-plane and the lower half-plane. When $G(z)$ is taken together with the upper half-plane one would expect it to be the Fourier transform of

$$f(t) = \begin{cases} 0 & (t > 0) \\ -1 & (t < 0) \end{cases}$$

and it is easy to check that this is indeed the case.

When $G(z)$ is taken together with the lower half-plane one would expect it to be the Fourier transform of

$$f(t) = \begin{cases} 1 & (t > 0) \\ 0 & (t < 0) \end{cases}$$

and again it is easy to check that this is the case.

8. If $a < \gamma_1 < \gamma_2 < b$ we consider for $R > 0$, $\int G(z)e^{izt}dz$ round the rectangle with vertices $-R + i\gamma_1$, $R + i\gamma_1$, $R + i\gamma_2$, $-R + i\gamma_2$. Then

$$\left|\int_{R+i\gamma_1}^{R+i\gamma_2} G(z)e^{izt}dz\right| \leq \frac{N}{R}.e^{(|\gamma_1|+|\gamma_2|).|t|}.(\gamma_2 - \gamma_1)$$

$$\to 0 \ (R \to \infty);$$

and a similar result holds for the integral over $[-R + i\gamma_1, -R + i\gamma_2]$.

Since the integrand is analytic within the rectangle we see that

$$\lim_{R\to\infty} \int_{-R+i\gamma_2}^{R+i\gamma_1} G(z)e^{izt}dz = \lim_{R\to\infty} \int_{-R\_i\gamma_2}^{R+i\gamma_2} G(z)e^{izt}dz.$$

9. One checks that $G(z)$ is properly defined for $\text{Im } z < b$ by noting that the integral converges. To prove analyticity we use the result of Theorem 4.2.3.

[Remark: The convergence of Definition 4.2.2 is usually described as local uniform convergence.]

Let K be a compact subset of $\{z\colon \text{Im } z < b\}$. Then there is some $\eta > 0$ so that $b - \text{Im } z \geq \eta$ for all $z \in K$. Set

$$f_n(z) = \int_0^n f(t)e^{izt}dt$$

and then for $m > n$ and $z \in K$,

$$\begin{aligned} |f_m(z) - f_n(z)| &= \left|\int_n^m f(t)e^{-izt}dt\right| \\ &\leq K \int_n^m e^{-bt}.e^{yt}dt \quad (y = \text{Im} z) \\ &\leq K \int_n^m e^{-\eta t}dt \\ &\leq \frac{K}{\eta}.e^{-\eta n} \end{aligned}$$

Given $\varepsilon > 0$ choose n_0 so that $\frac{K}{\eta} e^{-\eta n_0} < \varepsilon$ and then

$$|f_m(z) - f_n(z)| < \varepsilon \ (z \in K, \ m, \ n > n_0),$$

and so the sequence (f_n) converges uniformly on K. Hence the result follows from Theorem 4.2.3.

10. Note that here and later $u(t)$ is the unit step function, also known as the Heaviside function, defined in Example8.4.1 at the bottom of p.373. Note also that the next few problems are meant to illustrate the way in

which the theorems of §8.4 may be used. However, even if one or other of these theorems is applicable it may be easier to argue directly.

We have that

$$F[u(t).t^{\alpha}] = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} t^{\alpha} e^{-izt} dt.$$

If $z = x + iy$,

$$|e^{-izt}| = |e^{-ixt+yt}| = e^{yt},$$

and so for convergence of the integral we require $\operatorname{Im} z = y < 0$.

Suppose that $z = re^{i\theta}$ with $r > 0$ and $-\pi < \theta < 0$, so that $\operatorname{Im} z < 0$. Consider $\varsigma^{\alpha} e^{-iz\varsigma}$ integrated round the contour consisting of the real axis from 0 to $R > 0$, the arc of the circle $\{|\varsigma| = R\}$ from $\varsigma = R$ to $\varsigma = Re^{i(-\theta-\pi/2)}$ (note $-\pi < \theta < 0$) and the segment of the ray $\arg \varsigma = -\theta - \frac{\pi}{2}$ from $Re^{i(-\theta-\pi/2)}$ to the origin. If a $\alpha > 0$ is not an integer we would indent the contour at the origin. $\varsigma^{\alpha} = e^{\alpha \log \varsigma}$ is determined to be real on the positive real ς-axis and elsewhere within and on the contour by continuity.

Observing that $\varsigma^{\alpha} e^{-iz\varsigma}$ has no singularities within the contour and using the calculus of residues and letting $R \to \infty$ we find that

$$\begin{aligned}
\int_0^{\infty} t^{\alpha} e^{-izt} dt &= \int_0^{\infty} t^{\alpha} e^{i\alpha(-\theta-\pi/2)} . e^{-rt} . e^{i(-\theta-\pi/2)} dt \\
&= e^{i(\alpha+1)(-\theta-\pi/2)} \int_0^{\infty} t^{\alpha} e^{-rt} dt \\
&= e^{i(\alpha+1)(-\theta-\pi/2)} . \frac{1}{r^{\alpha+1}} \int_0^{\infty} u^{\alpha} e^{-u} du. \\
&= \frac{1}{(ire^{i\theta})^{\alpha+1}} \Gamma(\alpha+1) \\
&= \frac{\Gamma(\alpha+1)}{(iz)^{\alpha+1}}.
\end{aligned}$$

We have used that as $R \to \infty$ the integral along the arc of $\{|\varsigma| = R\}$ tends to 0 and that we can let the indentation, if one is necessary, shrink to 0.

11. We consider only

$$F[u(t)\cosh \omega t] = \frac{1}{\sqrt{2\pi}}\int_0^\infty \cosh \omega t . e^{-izt} dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{e^{\omega t} + e^{-\omega t}}{2} . e^{-izt} dt.$$

Now, assuming ω is real,

$$\int_0^\infty e^{(\omega - iz)t} dt = \frac{1}{\omega - iz},$$

$$\int_0^\infty e^{-(\omega + iz)t} dt = -\frac{1}{\omega + iz}$$

with $\operatorname{Im} z < -|\omega|$. Hence

$$F[u(t)\cosh \omega t] = \frac{1}{2\sqrt{2\pi}}\left(\frac{1}{\omega - iz} - \frac{1}{\omega + iz}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{iz}{\omega^2 + z^2}.$$

Remark: If we observe that $\cosh \omega t = \cos i\omega t$ we can use the result of Example 8.4.5 on p.374. However, one has to be careful about the associated strip of analyticity.

12. Here we can proceed as in the preceding problem.

13. The transform asked for is

$$\frac{1}{\sqrt{2\pi}}\int_\tau^\infty te^{at} . e^{-izt} dt = \frac{1}{\sqrt{2\pi}}\int_\tau^\infty te^{(a-iz)t} dt.$$

With the appropriate restriction on Im z the integral is easily evaluated using integration by parts.

14. One can deal with this problem in the same way as the preceding one.

15. The function $\frac{1}{z(z^2 - 1)(z^2 + 1)}$ has 5 singularities and these are at $z = 0, \pm 1, \pm i$. Considering the discussion in §8.3 if we consider this function as a Fourier transform there is an associated strip of analyticity which should emerge from the solution.

In partial fractions

$$\frac{1}{z(z^2 - 1)(z^2 + 1)} = -\frac{1}{z} + \frac{1}{2}\frac{z}{z^2 - 1} + \frac{1}{2}\frac{z}{z^2 + 1}$$

$$= -\frac{1}{z} + \frac{1}{4}\left(\frac{1}{1 + z} - \frac{1}{1 - z}\right) + \frac{1}{4i}\left(\frac{1}{1 - iz} - \frac{1}{1 + iz}\right).$$

If $\zeta = \xi + i\eta$, then

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{\zeta t-izt}dt = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{(\zeta-iz)t}dt$$

$$= -\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\zeta - iz},$$

provided $\xi + y < 0$, i.e. $\mathrm{Im}\ z < -\xi$ and so

$$F[u(t)e^{\zeta t}] = -\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\zeta - iz}$$

provided $\mathrm{Im}\ z < -\mathrm{Re}\ \zeta$. Hence

$$f(t) = -i\sqrt{2\pi}\ u(t) + \frac{1}{4}\ i\sqrt{2\pi}\ u(t)\ e^{-it} - \frac{1}{4}\ (-\ i\sqrt{2\pi}\ u(t)\ e^{it})$$

$$+ \frac{1}{4i}\ (-\ \sqrt{2\pi}\ u(t)\ e^{t}) - \frac{1}{4i}\ (\sqrt{2\pi}\ u(t)\ e^{-t})$$

$$= i\sqrt{2\pi}\ u(t)\ [-\ 1 + \frac{e^{-it}}{4} + \frac{e^{it}}{4} - \frac{1}{4}\ e^{t} + \frac{1}{4}\ e^{-t}]$$

$$= -i\sqrt{2\pi}\ u(t)\ [1 - \frac{1}{2}\cos t + \frac{1}{2}\sinh t].$$

16. From the discussion at the beginning of §8.5 one finds that

$$F[y] = \frac{F[e^{-|t|}]}{(iz)^2 + 3(iz) + 2}$$

$$= \frac{F[e^{-|t|}]}{(1 + iz)(2 + iz)}.$$

From solution to P.5,

$$F[e^{-|t|}] = \frac{1}{\sqrt{2\pi}}\ (\frac{1}{1 + iz} + \frac{1}{1 - iz})$$

and so

$$F[y] = \frac{1}{\sqrt{2\pi}}\ (\frac{1}{1 + iz} + \frac{1}{1 - iz})(\frac{1}{1 + iz} - \frac{1}{2 + iz})$$

$$= \frac{1}{\sqrt{2\pi}}\ [\frac{1}{(1 + iz)^2} + \frac{1}{2}\ (\frac{1}{1 + iz} + \frac{1}{1 - iz})$$

$$-\ (\frac{1}{1 + iz} - \frac{1}{2 + iz}) - \frac{1}{3}\ (\frac{1}{1 - iz} + \frac{1}{2 + iz})]$$

$$= \frac{1}{\sqrt{2\pi}}\ [\frac{1}{(1 + iz)^2} - \frac{1}{2}\cdot\frac{1}{1 + iz} + \frac{1}{6}\cdot\frac{1}{1 - iz} + \frac{2}{3}\cdot\frac{1}{2 + iz}].$$

Hence

$$y(t) = [(t - \frac{1}{2})e^{-t} + \frac{2}{3}\ e^{-2t}]\ u(t) + \frac{1}{6}\ e^{t}\ [1 - u(t)].$$

It is easily checked that this is a solution of the equation for $t > 0$ and for $t < 0$.

17. From Example 8.3.1, p.366,

$$F[u(t)\ \sin \omega t] = \frac{1}{\sqrt{2\pi}}\ \frac{\omega}{\omega^2 - z^2}\ (\mathrm{Im}\ z < 0).$$

Hence

$$F[y] = -\frac{1}{\sqrt{2\pi}}\cdot\frac{\omega}{\omega^2 - z^2}\cdot\frac{1}{z^2 - 3iz - 2}.$$

One can now proceed as in the preceding solution or one can argue directly and we shall do the latter.

The relevant strip is the lower half-plane so we consider

$$y(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty+i\gamma}^{\infty+i\gamma} -\frac{1}{\sqrt{2\pi}}\cdot\frac{\omega}{\omega^2 - z^2}\cdot\frac{e^{izt}}{z^2 - 3iz - 2}\,dz$$

$$= -\frac{\omega}{2\pi}\int_{-\infty+i\gamma}^{\infty+i\gamma} \frac{e^{izt}}{(\omega^2 - z^2)(z^2 - 3iz - 2)}\,dz,$$

where $\gamma < 0$. Using the contour of Figure 8.3.1 on p.366 and arguing in the usual way we find that for $t > 0$,

$$\int_{-\infty+i\gamma}^{\infty+i\gamma} \frac{e^{izt}}{(\omega^2 - z^2)(z^2 - 3iz - 2)}\,dz = 2\pi i \times \text{sum of residues at } \pm\omega,\ i,\ \text{and } 2i$$

$$= 2\pi i\left[\frac{e^{i\omega t}}{(-2\omega)(\omega^2-3i\omega-2)} + \frac{e^{-i\omega t}}{2\omega(\omega^2+3i\omega-2)} + \frac{e^{-t}}{(\omega^2+1)(-i)} + \frac{e^{-2t}}{(\omega^2+4)(i)}\right]$$

$$= \frac{\pi i}{\omega}\left[\frac{e^{-i\omega t}}{\omega^2+3i\omega-2} - \frac{e^{i\omega t}}{\omega^2-3i\omega-2}\right] + 2\pi\left[-\frac{e^{-t}}{\omega^2+1} + \frac{e^{-2t}}{\omega^2+4}\right]$$

$$= \frac{\pi i}{\omega}\left[\frac{(\omega^2-2)(e^{-i\omega t}-e^{i\omega t})-3i\omega(e^{-i\omega t}+e^{i\omega t})}{(\omega^2-2)^2+9\omega^2}\right] + 2\pi\left[-\frac{e^{-t}}{\omega^2+1} + \frac{e^{-2t}}{\omega^2+4}\right]$$

$$= \frac{\pi i}{\omega}\left[\frac{-2i(\omega^2-2)\sin\omega t-6i\omega\cos\omega t}{\omega^4+5\omega^2+4}\right] + 2\pi\left[-\frac{e^{-t}}{\omega^2+1} + \frac{e^{-2t}}{\omega^2+4}\right]$$

$$= \frac{2\pi}{\omega}\left[\frac{(\omega^2-2)\sin\omega t+3\omega\cos\omega t}{\omega^4+5\omega^2+4}\right] + 2\pi\left[-\frac{e^{-t}}{\omega^2+1} + \frac{e^{-2t}}{\omega^2+4}\right].$$

Hence for $t > 0$,

$$y(t) = \omega\left[\frac{e^{-t}}{\omega^2+1} - \frac{e^{-2t}}{\omega^2+4}\right] - \left[\frac{(\omega^2-2)\sin\omega t+3\omega\cos\omega t}{\omega^4+5\omega^2+4}\right].$$

The general solution of the homogeneous equation is

$$y(t) = Ae^{-t} + Be^{-2t}$$

and so the general solution of the given equation for $t > 0$ is

$$y(t) = Ae^{-t} + Be^{-2t} - \left[\frac{(\omega^2-2)\sin\omega t+3\omega\cos\omega t}{\omega^4+5\omega^2+4}\right].$$

[Remark: Had we noted the form of the general solution of the homogeneous equation earlier we could have dropped the terms involving e^{-t}, e^{-2t}.]

To find the solution of the problem we determine A, B from $y(t) \to 0$ $(t \to 0+)$, $y'(t) \to 0$ $(t \to 0+)$ and hence

$$A = \frac{\omega(\omega^2+4)}{\omega^4+5\omega^2+4}, \quad B = -\frac{\omega(\omega^2+1)}{\omega^4+5\omega^2+4}$$

18. If we apply the Fourier transform to both sides of the equation we get

$$[(iz) + \frac{1}{(iz)}]\, F[y] = F[u(t)\, e^{-t}],$$

where the second term on the left is the result of an appropriate integration by parts. Hence

$$F[y\} = \frac{1}{\sqrt{2\pi}} \frac{iz}{(1+iz)(1-z^2)}$$

$$= \frac{1}{\sqrt{2\pi}} [-\frac{1}{2}\cdot\frac{1}{iz+1} + \frac{1+i}{4}\cdot\frac{1}{iz+i} + \frac{1-i}{4}\cdot\frac{1}{iz-i}]$$

and so

$$y(t) = -\frac{1}{2} e^{-t} + \frac{1+i}{4} e^{-it} + \frac{1-i}{4} e^{it}$$

$$= \frac{1}{2}[-e^{-t} + \cos t + \sin t].$$

Considering the homogeneous equation we expect that it can be differentiated to give

$$\frac{d^2y}{dt^2} + y(t) = 0$$

which has solutions of the form A cos t + B sin t. Considering $y(t) = \cos t$, $y(t) = \sin t$ in

$$\frac{dy}{dt} + \int_0^t y(\tau)d\tau = 0$$

we see that a solution is of the form A cos t and so we are looking for a solution of the original problem of the form

$$y(t) = -\frac{1}{2} e^{-t} + \frac{1}{2} \sin t + A \cos t.$$

If $y(t) \to y_0 (t \to 0)$, then

$$y_0 = -\frac{1}{2} + A, \quad A = y_0 + \frac{1}{2}$$

and a solution of the problem is

$$y(t) = -\frac{1}{2}e^{-t} + \frac{1}{2}\sin t + (y_0 + \frac{1}{2})\cos t.$$

The next 3 problems seem rather difficult and are not comparable in many ways to the illustrative examples of § 8.6. We shall, therefore, be concerned in the following solutions with a general discussion of the problems rather than with finding explicit solutions in all cases.

19. Following the argument of Example 8.6.5 we are solving the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} - bx(\pi - x) \quad (0 < x < \pi,\ t > 0)$$

subject to the conditions

$$u(x, 0) = 0,\ u_t(x, 0) = 0;$$

$$u(0, t) = 0,\ u(\pi, t) = 0.$$

If we consider Im z < 0 and

$$U = U(x, z) = F[u] = \frac{1}{\sqrt{2\pi}}\int_0^\infty u(x, t)\, e^{-izt}dt,$$

then

$$U_{xx} + \frac{z^2}{a}U = -\frac{b}{\sqrt{2\pi}\,.iz}\,x(\pi - x).$$

The general solution of the homogeneous equation is

$$U(x,z) = A\sin\frac{zx}{a} + B\cos\frac{zx}{A},$$

where A, B are functions of z. A particular solution of the above equation is $c_0 + c_1x + c_2x^2$ with

$$2c_2 + \frac{z^2}{a^2}(c_0 + c_1x + c_2x^2) = -\frac{b}{\sqrt{2\pi}\,.iz}\,x(\pi - x)$$

so that

$$\frac{z^2}{a^2}c_2 = \frac{b}{\sqrt{2\pi}\,.iz}, \quad c_2 = \frac{a^2b}{\sqrt{2\pi}\,.iz^3},$$

$$\frac{z^2}{a^2}c_1 = -\frac{b\pi}{\sqrt{2\pi}\,.iz}, \quad c_1 = \frac{a^2b\pi}{\sqrt{2\pi}\,.iz^3},$$

$$2c_2 + \frac{z^2}{a^2}c_0 = 0, \quad c_0 = -\frac{2a^2}{z^2}c_2 = -\frac{2a^4b}{\sqrt{2\pi}\,.iz^5}$$

Hence the solution is

$$U(x, z) = A(z).\sin \frac{zx}{a} + B(z) \cos \frac{zx}{a}$$

$$- \frac{a^2 b}{\sqrt{2\pi}.iz^3} (\frac{2a^2}{z^2} + \pi x - x^2).$$

From the boundary conditions we can determine A and B and then use the inversion formula.

20. This problem involves solving

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2} \qquad (t > 0)$$

subject to the conditions

$$u(x, 0) = 0, \quad u_t(x, 0) = 0, \quad u(0, t) = g(t).$$

Because of the first two conditions one might consider taking a one-sided Fourier transform with respect to t. However, if one does this one arrives at an impasse because one is unable to determine a function A(z) in the general solution by the given conditions. In fact it seems necessary to use the Fourier sine transform as in Example 8.6.4 on p.384. Consider

$$U = U(z, t) = - \sqrt{\frac{2}{\pi}}\, i \int_0^\infty u(x, t) \sin zx\, dx.$$

Then, as on p.385,

$$U_{tt} + a^2 z^2 U = - \sqrt{\frac{2}{\pi}}.iza^2\, g(t)$$

and so, using the result on p.385,

$$U(z, t) = A \sin azt + B \cos azt - \sqrt{\frac{2}{\pi}}\, iza^2 \int_0^t g(\tau)\, e^{-a^2 z^2 (t-\tau)} d\tau .$$

The condition $u(x, 0) = 0$ requires that $B \equiv 0$, and the conditions $u(x, 0) = 0$, $u(0, t) = 0$ imply that $g(0) = \lim_{t\to 0} g(t) = 0$ so that $U_t = Aaz$ when $t = 0$ and therefore $A \equiv 0$. Therefore as on pp.385, 386 we find that

$$u(x, t) = \frac{x}{2a\sqrt{\pi}} \int_0^t \frac{g(\tau)\, \exp[\frac{x^2}{4a^2 (t-\tau)}]}{(t-\tau)^{3/2}}\, d\tau .$$

[Remark: Verification of this solution is really rather tricky.]

21. If one follows the hint then one is concerned with a doubly infinite string whose displacement satisfies the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2}.$$

This means that the extended problem must be one in which the string is freely vibrating for $t > 0$. This assumption is physically plausible given the hypotheses of the problem.

We consider Example 8.6.1 with $f(x)$ of the example replaced by $f(x)$ $(x > 0)$, $-f(-x)$ $(x < 0)$ and $g(x) \equiv 0$. The solution then, from p.382, is

$$u(x, t) = \frac{1}{2}[f(x + at) + f(x - at)],$$

where is the function of the problem extended to an odd function.

For the final part of the problem consider, as in Example 8.6.4,

$$u(x, t) = u_1(x, t) + u_2(x, t).$$

Since $f(0)$ denotes a displacement of the whole string up or down we can take $f(0) = 0$ and consider

$$u_1(x, 0)=f(x), \frac{\partial}{\partial t} u_1(x, 0) = 0, u_1(0, t) = 0 \text{ (i.e. the problem above);}$$

$$u_2(x, 0) = 0, \frac{\partial}{\partial t} u_2(x, 0) = 0, u_2(0, t) = g(t) \text{ (i.e. P.20).}$$

Chapter 9

Laplace Transforms

| READ § 9.1 |

The Laplace transform is regarded as a two-sided transform of functions which are 0 for $t < 0$. Whilst this means that one can use the results of Fourier transforms rather directly it might have been better to give an account that restricted attention to the one-sided case as this is an essential feature of the Laplace transform.

Definition 9.1.1 The transform $H(z) = L(f)$ will be analytic for $\text{Re } z > b$.

Theorem 9.1.1 A Laplace transform $H(z)$ is always analytic for all z of large enough real part and so the inversion formula will be valid for all large enough positive γ.

Self-Assessment Questions

SAQ 1, 2, 3 are Exercises 9.1, #1, 2, 3.

| READ § 9.2 |

One should not be too concerned about the proofs of the theorems in this section. If one proceeds in a formal manner one arrives at the conclusions of these theorems. In Theorem 9.2.2, for example, we have, assuming that the symbols have a meaningful interpretation,

$$L(f) = \int_0^\infty f(t)\, e^{-zt} dt = - \frac{f(t)e^{-zt}}{z}\Big|_0^\infty + \frac{1}{z}\int_0^\infty f'(t)\, e^{-zt}\, dt$$

$$= \frac{f(0)}{z} + \frac{1}{z} L[f'(t)],$$

and this is the formula of the theorem. In any special case it should be clear whether the steps above are valid and the result true in the special case.

Example 9.2.8 The notation used in this example is not in conformity with more or less universal practice and so is very misleading. If one has a derivative $f^{(n)}(t)$ then $f^{(n)}(kt)$, as one would expect, is the value of the nth derivative at kt, i.e. we differentiate n times then give the variable the value kt. We do NOT give the variable of f the value kt and then differentiate n times with respect to t. In other words

$$f^{(n)}(kt) = \frac{d^n}{dx^n} f(x) \Big|_{x=kt} \quad \text{and } \underline{\text{NOT}} \ \frac{d^n}{dt^n} f(kt).$$

However, Dettman uses the latter notation so that

$$t \, J_0''(kt) + J_0'(kt) + k^3 t \, J_0(kt) = 0$$

does NOT mean what one expects, but

$$t \frac{d^2}{dt^2} J_0(kt) + \frac{d}{dt} J_0(kt) + k^2 t \, J_0(kt) = 0.$$

Perhaps it is also worth noting that one could take $k = 1$ initially and then find $L(J_0(kt)]$ by a transformation of the variable in $L[J_0(t)]$. Note also that $J_0(-t) = J_0(t)$ and therefore we can assume that $k > 0$ without loss of generality.

Self-Assessment Question

SAQ 4. is Exercises 9.2, #1.

| READ § 9.3 |

Theorem 9.3.1 In the displayed formula of the proof the minus signs should be plus signs.

Theorem 9.3.2 et seq. One should read these theorems in order to get an idea of how one deals with the inversion problem. It is normally better, provided one has a grasp of the general ideas of these theorems, to treat a particular problem on its own merits.

Problems for § 9.1, § 9.2, § 9.3

P1, 2, 3 are Exercises § 9.2, #2, 3, 4.

P4, 5, 6 are Exercises § 9.3, #1, 3, 4.

| READ § 9.4 |

Note carefully the general method for using Laplace transforms in order to solve ordinary differential equations. The examples are rather complicated and you should not spend too much time on the details unless you are interested in the physics context. Observe how the Laplace transform copes with initial conditions.

Problems for §9.4

P7, 8, 9, 10 are Exercises 9.4, #1, 2, 3, 4.

§9.5 is a special topic and does not form part of M828.

| READ §9.6 |

Note that this section does not contain anything that is essentially new. As the introduction says, this section gives the Laplace transform version of what was done in §8.6 using the Fourier transform.

Problems for §9.6

P11, 12, 13, 14 are Exercises 9.6, #1, 2, 3, 4.

| READ §9.7 |

Problems for §9.7

P15, 16, 17 are Exercises 9.7, #1, 2, 3.

Solutions to Self-Assessment Questions

1. We have

$$L[u(t)] = \int_0^\infty e^{-zt}dt = -\frac{1}{z}\Big|_0^\infty = \frac{1}{z} \quad (\text{Re } z > 0).$$

The verification of the inversion formula requires us to think of L[u(t)] as a two-sided transform and so we are concerned with the formula at the bottom of p.399. Hence the inverse transform is

$$\frac{1}{2\pi i}\int_{\gamma = i\infty}^{\gamma + i\infty} \frac{e^{\zeta t}}{\zeta}\, d\zeta \quad (\text{Re } \gamma > 0).$$

If $\zeta = \xi + i\eta$, then

$$|e^{\zeta t}| = e^{\xi t}$$

and so for t > 0 we consider $\frac{1}{2\pi i}\int_{\gamma-iR}^{\gamma+iR}$ together with the integral over the semi-circle from $\gamma + iR$ to $\gamma - iR$ on the left. As $R \to \infty$, the integral over the semi-circle tends to 0 and so

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\zeta t}}{\zeta}\, d\zeta = \operatorname{Res} \frac{e^{\zeta t}}{\zeta}\Big|_{\zeta=0} = 1$$

If t < 0 we use a semi-circle on the right and note that in the contour the integrand has no singularities. Hence

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\zeta t}}{\zeta}\, d\zeta = 0.$$

When t = 0 we have $\frac{1}{2\pi i}\int_{\gamma=i\infty}^{\gamma+i\infty} \frac{d\zeta}{\zeta}$ which does not converge. However, if one considers Theorem 8.2.1 and its proof one sees that it is the Cauchy principal value that is relevant. Now

$$\frac{1}{2\pi i}\int_{\gamma-iR}^{\gamma+iR} \frac{d\zeta}{\zeta} = \frac{1}{2\pi i}[\log(\gamma+iR) - \log(\gamma-iR)],$$

where we determine the logarithms to be real for R = 0 and continuous in R. Then

$$\operatorname{Im}[\log(\gamma+iR)] \to \pi/2 (R \to \infty), \quad \operatorname{Im}[\log(\gamma-iR)] \to -\pi/2 (R \to \infty)$$

and

$$\log \left|\frac{\gamma+iR}{\gamma-iR}\right| \to 0 \quad (R \to \infty)$$

and so

$$\frac{1}{2\pi i}\int_{\gamma-iR}^{\gamma+iR} \frac{d\zeta}{\zeta} \to \frac{1}{2\pi i}\left[\frac{i\pi}{2} - \left(-\frac{i\pi}{2}\right)\right] = \frac{1}{2}$$

$$= \frac{u(0^+) + u(0^-)}{2}.$$

2. Assuming ω is real we have

$$L[u(t)\cos \omega t] = \int_0^\infty \cos \omega t . e^{-zt} dt \quad (\operatorname{Re} z > 0)$$

$$= \int_0^\infty \frac{e^{i\omega t} + e^{-i\omega t}}{2} . e^{-zt} dt$$

$$= \frac{1}{2}\left[\frac{e^{(i\omega-z)t}}{i\omega-z} - \frac{e^{-(i\omega+z)t}}{i\omega+z}\right]\Big|_0^\infty$$

$$= \frac{1}{2}\left[\frac{1}{i\omega+z} - \frac{1}{i\omega-z}\right]$$

$$= \frac{z}{z^2+\omega^2}.$$

For $t > 0$ we consider the usual semi-circle on the left from $\gamma+iR$ to $\gamma-iR$ and obtain, for $\gamma > 0$,

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{z}{z^2+\omega^2}\, e^{zt}dz = \text{Res}\, \frac{ze^{zt}}{z^2+\omega^2}\bigg|_{z+i\omega} + \text{Re}\, \frac{ze^{zt}}{z^2+\omega^2}\bigg|_{z=-i\infty}$$

$$= \frac{i\omega e^{i\omega t}}{2i\omega} - \frac{i\omega e^{-i\omega t}}{-2i\omega}$$

$$= \frac{e^{i\omega t} + e^{-i\omega t}}{2}$$

$$= \cos \omega t.$$

The cases $t < 0$, $t = 0$ are dealt with in the standard way.

3. Assuming that differentiation under the integral sign is permissible

$$-\frac{d}{dz} L[f(t)] = -\frac{d}{dz}\int_0^\infty f(t)\, e^{-zt}dt$$

$$= -\int_0^\infty f(t)\, \frac{\partial}{\partial z}\, [e^{-zt}].dt$$

$$= \int_0^\infty t\, f(t)\, e^{-zt}dt$$

$$= L[t\, f(t)].$$

From Example 9.1.1,

$$L(u(t) \sin \omega t] = \frac{\omega}{z^2+\omega^2}$$

and hence

$$L[t\, u(t) \sin \omega t] = -\frac{d}{dz}\left[\frac{\omega}{z^2+\omega^2}\right]$$

$$= \frac{2z\omega}{(z^2+\omega^2)^2}.$$

4.(a) Assuming a is real,

$$L[u(t)\, t^n e^{at}] = \int_0^\infty t^n e^{at} e^{-zt} dt \qquad (\text{Re } z > a)$$

$$= \int_0^\infty t^n e^{-(z-a)t} dt.$$

Now suppose that $z = x$ is real and change the variable of integration by $(x - a)t = \tau$ and then

$$L[u(t)\ t^n e^{at}] = \frac{1}{(x-a)^{n+1}} \int_0^\infty \tau^n e^{-\tau}\, d\tau$$

$$= \frac{n!}{(x-a)^{n+1}}.$$

By analytic continuation the transform is $\dfrac{n!}{(z-a)^{n+1}}$ (Re $z > a$).

(b) Assuming ω is real,

$$L[u(t) \cos h\, \omega t] = \int_0^\infty \cos h\, \omega t\ e^{-zt} dt$$

$$= \frac{1}{2} \int_0^\infty (e^{\omega t} + e^{-\omega t}).e^{-zt} dt \qquad (\text{Re } z > |\omega|)$$

$$= \frac{1}{2} \left[\frac{e^{(\omega-z)t}}{\omega-z} - \frac{e^{-(\omega+z)t}}{\omega+z}\right] \Big|_0^\infty$$

$$= \frac{1}{2} \left[\frac{1}{\omega+z} - \frac{1}{\omega-z}\right]$$

$$= \frac{z}{z^2-\omega^2}.$$

(c) Assuming b is real and a > 0,

$$L[t\ u(t-a)e^{bt}] = \int_0^\infty t\ u(t-a)e^{-(z-b)t} dt \qquad (\text{Re } z > b)$$

$$= \int_a^\infty te^{-(z-b)t} \Big|_a^\infty + \frac{1}{z-b} \int_a^\infty e^{-(z-b)t} dt$$

$$= -\frac{t}{z-b}.e^{-(z-b)t} \Big|_a^\infty + \frac{1}{z-b} \int_a^\infty e^{-(z-b)t} dt$$

$$= \frac{a}{z-b}\, e^{-(z-b)a} - \frac{1}{(z-b)^2}\, e^{-(z-b)t} \Big|_a^\infty$$

$$= \frac{a}{z-b}.e^{-(z-b)a} + \frac{1}{(z-b)^2}\, e^{-(z-b)a}$$

$$= \frac{e^{ab}.e^{-az}}{(z-b)^2} + \frac{ae^{ab}.e^{-az}}{z-b}.$$

(d) The function $\int_0^t e^{-\tau} \cos \omega(t-\tau) d\tau$ is the convolution of e^{-t} and $\cos \omega t$.

The one theorem it is perhaps useful to keep in mind from §9.2 is Theorem 9.2.7 which deals with convolutions. From this theorem the transform asked for is

$$L[u(t)\ e^{-t}].L[u(t)\cos \omega t] = \int_0^\infty e^{-(z+1)t}dt.\int_0^\infty \cos \omega t.e^{-zt}dt$$

$$= \frac{1}{z+1}.\frac{z}{z^2+\omega^2}.$$

[The answer given by Dettman is wrong.]

Solutions to Problems

1. For $x > b$, $\delta > 0$ consider

$$I_\delta(x) = \int_x^\infty dt \int_\delta^\infty f(u)\ e^{-tu}du.$$

The hypotheses of the problem ensure that $I_\delta(x)$ exists for $x > b$ and $\delta > 0$ and it can be shown that for such x, δ the orders of integration can be interchanged. Hence

$$I_\delta(x) = \int_\delta^\infty f(u)du \int_x^\infty e^{-tu}dt$$

$$= \int_\delta^\infty \frac{f(u)}{u}.e^{-xu}du.$$

The condition on $\frac{f(t)}{t}$ as $t \to 0$ shows that

$$I_0(x) = \lim_{\delta\to 0} I_\delta(x) = \int_0^\infty \frac{f(u)}{u}.e^{-xu}du = L[\frac{f(u)}{u}].$$

In the integral for $I_\delta(x)$, given the hypotheses of the problem, one can prove that $\lim\limits_{\delta\to 0}$ can be interchanged with the integration with respect to t and hence

$$I_0(x) = \int_x^\infty G(t)dt = H(x).$$

Consequently $H(z)$ and $L[\frac{f(u)}{u}]$ as a function of z are equal for each real $z > b$ and so the result follows by analytic continuation.

2.(a) Let ω be real and put

$$H(z) = \int_0^\infty \frac{\sin \omega t}{t}\ e^{-zt}dt \qquad (\text{Re } z > 0).$$

Assuming that differentiation under the integral sign is justified (which it is),

$$H'(z) = -\int_0^\infty \sin \omega t.e^{-zt} dt \qquad (\text{Re } z > 0)$$

$$= -\frac{\omega}{z^2+\omega^2}.$$

Considering $z = x > 0$ and noting that $H(\infty) = \lim_{x\to\infty} H(x) = 0$,

$$-H(x) = \int_x^\infty H'(t)\,dt = \int_x^\infty \frac{\omega}{t^2+\omega^2} dt$$

$$= -\tan^{-1}\left(\frac{\omega}{x}\right).$$

By continuation

$$H(z) = \tan^{-1}\left(\frac{\omega}{z}\right) \qquad (\text{Re } z > 0).$$

(b) The proof is similar to that of (a).

(c) Let

$$H(z) = \int_0^\infty \frac{J_1(t)}{t}.e^{-zt} dt \qquad (\text{Re } z > 0).$$

For convergence at the upper limit consider the representation for $J_1(t)$ on p.316 and for convergence at the lower limit consider the series for $J_1(t)$ on p.312. These results can also be shown to justify differentiation under the integral sign and so

$$H'(z) = -\int_0^\infty J_1(t)e^{-zt} dt = -\left[1 - \frac{z}{\sqrt{1+z^2}}\right],$$

from the result at the top of p.410.

Taking $x > 0$ and integrating from x to ∞ and noting that $H(R) \to 0$ ($R \to \infty$, R real) we obtain

$$-H(x) = -\lim_{R\to\infty} \int_x^R \left[1 - \frac{t}{\sqrt{1+t^2}}\right] dt$$

$$= -\lim_{R\to\infty} [R - x - (\sqrt{1+R^2} - \sqrt{1+x^2})].$$

Since

$$\lim_{R\to\infty} [R - \sqrt{1+R^2}] = \lim_{R\to\infty} \left[-\frac{1}{R+\sqrt{1+R^2}}\right] = 0,$$

we deduce that

$$H(x) = \sqrt{1+x^2} - x.$$

In the integral for H(x) put t = ku with k > 0 and then

$$H(x) = \int_0^\infty \frac{J_1(ku)}{ku}.e^{-xku}.k\,du$$

$$= \int_0^\infty \frac{J_1(ku)}{u}.e^{-xku}du.$$

From this result and analytic continuation, for Re z > 0,

$$L[\frac{J_1(kt)}{t}] = H(\frac{z}{k}) = \frac{1}{k}[\sqrt{(k^2 + z^2)} - z].$$

Since $J_1(-kt) = -J_1(kt)$ this result is also valid for k < 0 and letting $k \to 0$ one checks the result for k = 0.

Note that $\sqrt{(k^2 + z^2)}$ in {Re z > 0} is positive on the positive real axis and is determined elsewhere by continuity.

3. The reference to Theorem 9.1.1 seems to be a mistake. We suppose that L[f] exists and that Re z > 0. Let N be a positive integer and then

$$L[f] = \lim_{N\to\infty} \int_0^{N\tau} f(t)\, e^{-zt}\, dt$$

$$= \lim_{N\to\infty} \sum_{k=0}^{N-1} \int_{k\tau}^{(k+1)\tau} f(t)e^{-zt}dt.$$

In the kth term of the sum put $t = k\tau + u$ and note that by periodicity $f(k\tau + u) = f(u)$. Hence

$$\int_{k\tau}^{(k+1)\tau} f(t)e^{-zt}dt = e^{-zk\tau} \int_0^\tau f(u)e^{-zu}du$$

and so

$$\int_0^{N\tau} f(t)e^{-zt}dt = [\sum_{k=0}^{N-1} e^{-zk\tau}] \int_0^\tau f(u)e^{-zu}du$$

$$= \frac{1 - e^{-z\tau N}}{1 - e^{-z\tau}}.\int_0^\tau f(u)e^{-zu}du.$$

Since Re z > 0, $e^{-z\tau N} \to 0$ $(N \to \infty)$ and therefore, letting $N \to \infty$,

$$L[f] = \frac{1}{1 - e^{-z\tau}} \int_0^\tau f(u)e^{-zu}du.$$

One would expect the rider to be a special case of the general result, but it is not. What would seem a more natural choice is the fractional part of t, i.e. {t} = t - [t]. However, for the question asked,

$[t] = n(n \le t < n + 1)$

and so

$$\int_0^\infty [t]e^{-zt}dt = \sum_{n=0}^{\infty} n \int_n^{n+1} e^{-zt}dt \qquad (\mathrm{Re}\ z > 0)$$

$$= \sum_{n=0}^{\infty} \frac{n}{z} [e^{-zn} - e^{-z(n+1)}]$$

$$= \frac{1 - e^{-z}}{z} \sum_{n=0}^{\infty} ne^{-zn}$$

$$= \frac{1 - e^{-z}}{z} . - \frac{d}{dz} [\sum_{n=0}^{\infty} e^{-zn}]$$

$$= \frac{1 - e^{-z}}{z} . - \frac{d}{dz} [\frac{1}{1 - e^{-z}}]$$

$$= \frac{1}{z(1 - e^{-z})}.$$

4. This problem seems unduly elaborate for an exercise that is meant to illustrate the general ideas and results. One must find the partial fraction form of the given function. Since

$$z^2 + 2z + 5 = (z + 1)^2 + 2^2 = (z + 1 + 2i)(z + 1 - 2i)$$

we consider

$$\frac{z}{(z+1)(z-2)^2(z^2+2z+5)} = \frac{A}{z+1} + \frac{B}{z-2} + \frac{C}{(z-2)^2} + \frac{D}{z+1+2i} + \frac{E}{z+1-2i}.$$

Multiply by z+1 and put z=-1 and then

$$A = \frac{-1}{(-1-2)^2[(-1)^2+2.(-1)+5]}$$

$$= \frac{-1}{9(1-2+5)}$$

$$= -\frac{1}{36}.$$

One can determine D, E similarly. Multiplying by $(z - 2)^2$ and putting $z = 2$ determines C. Finally B can be determined by multiplying by z and letting $z \to \infty$ to obtain

$$0 = A + B + D + E, \quad B = - A - D - E.$$

The relevant function can then be found as in Example 9.3.1 on p.412.

5. We shall verify entry 13 only. Because §9.3 deals with inversion in the manner it does and G(z) is on the left of the table you might think that you are required to start with G(z) and find f(t). However, in a verification one should proceed in the simplest possible way and so in this case one should start with f(t) and find G(z).

If

$$H(z) = \int_0^\infty \frac{1 - \cos \omega t}{t} . e^{-zt} dt \qquad (\text{Re } z > 0),$$

then

$$H'(z) = - \int_0^\infty (1 - \cos \omega t) e^{-zt} dt$$

$$= - \left[\frac{1}{z} + \frac{z}{z^2 + \omega^2}\right].$$

Considering $z = x > 0$ and noting that $H(R) \to 0$ $(R \to \infty)$ we get

$$- H(x) = \lim_{R\to\infty} \int_x^\infty H'(u)\, du$$

$$= - \frac{1}{2} [\log(x^2 + \omega^2) - \log x^2]$$

$$+ \lim_{R\to\infty} \frac{1}{2} \log \frac{R^2 + \omega^2}{R^2}$$

and so

$$H(x) = \frac{1}{2} [\log(x^2 + \omega^2) - \log x^2].$$

The result now follows by analytic continuation.,

6. Note that b_1, b_2 should be B_1 , B_2. A cubic polynomial can be put in the form

$$\alpha z\{(z - a)^2 + b^2\} + \beta\{(z - a)^2 + b^2\} + \gamma z + \delta$$

and hence one obtains the given form of $\frac{P(z)}{Q(z)}$ with $A_2 = 0$.

To find f(t) so that

$$L[f(t)] = \frac{P(z)}{Q(z)}$$

we use 8 and 9 of the table on p.415 and the results obtained from 8 and 9 by differentiating with respect to z.

7. If $Y(z) = L[y(t)]$, then

$$\int_0^\infty y''(t) e^{-zt} dt + 3 \int_0^\infty y'(t) e^{-zt} dt + 2\, Y(z) = \int_0^\infty e^{-(z+1)t} dt.$$

Now

$$\int_0^\infty y'(t) e^{-zt} dt = y(t) e^{-zt} \Big|_0^\infty + z \int_0^\infty y(t) e^{-zt} dt$$

$$= - y_0 + z\, Y(z);$$

$$\int_0^\infty y''(t)e^{-zt}dt = y'(t)e^{-zt}\Big|_0^\infty + z\int_0^\infty y'(t)e^{-zt}dt$$

$$= - y_0' - y_0 z + z^2\ Y(z).$$

Hence

$$z^2\ Y(z) - y_0 z - y_0' + 3z\ Y(z) - 3\ y_0 + 2\ Y(z) = \frac{1}{z+1}$$

and so

$$Y(z) = \frac{1/z+1 + y_0 z + y_0' + 3y_0}{z^2 + 3z + 2}$$

$$= \frac{1}{(z+1)(z+1)(z+2)} + \frac{y_0 z + y_0' + 3y_0}{(z+1)(z+2)}.$$

We now determine y(t) from the inverse Laplace transform.

8. If $Y(z) = L[y(t)]$ then

$$\int_0^\infty y''(t)e^{-zt}dt + \omega_0^2\ Y(z) = \int_0^\infty \sin\omega t.e^{-zt}dt.$$

Now

$$\int_0^\infty y''(t)e^{-zt}dt = y'(t)e^{-zt}\Big|_0^\infty + z\int_0^\infty y'(t)\ e^{-zt}dt$$

$$= z\ y(t)e^{-zt}\Big|_0^\infty + z^2\ Y(z)$$

$$= z^2\ Y(z);$$

and

$$\int_0^\infty \sin\omega t.e^{-zt}dt = \frac{\omega}{z^2+\omega^2}.$$

Hence

$$Y(z) = \frac{\omega}{(z^2+\omega_0^2)(z^2+\omega^2)}.$$

For $\omega \neq \omega_0$ we use 4 of the table on p.415 and when $\omega = \omega_0$ we consider the differentiated forms of 4 and 5.

9. Let $Y(z) = L[y(t)]$ and consider

$$\int_0^\infty y'(t)e^{-zt}dt + \int_0^\infty \left[\int_0^t y(\tau)d\tau\right] e^{-zt}dt = \int_0^\infty e^{-(z+1)t}dt.$$

Now

$$\int_0^\infty y'(t)e^{-zt}dt = y(t)e^{-zt}\Big|_0^\infty + z\,Y(z)$$

$$= -\,y_0 + z\,Y(z);$$

$$\int_0^\infty \Big[\int_0^t y(\tau)d\tau\Big]\,e^{-zt}dt = -\frac{1}{z}\Big[\int_0^t y(\tau)d\tau\Big].e^{-zt}\Big|_0^\infty + \frac{1}{z}\,Y(z)$$

$$= \frac{1}{z}\,Y(z).$$

Hence

$$-\,y_0 + z\,Y(z) + \frac{1}{z}\,Y(z) = \frac{1}{z+1}$$

and so

$$Y(z) = \frac{z(z+1+y_0)}{z^2+1}.$$

One now use the inverse Laplace transform to find y(t).

10. Let $X(z) = L[x(t)]$, $Y(z) = L[y(t)]$ and consider

$$\int_0^\infty x'(t)e^{-zt}dt + \int_0^\infty y'(t)e^{-zt}dt - X(z) + 3Y(z) = \frac{1}{z+1},$$

$$\int_0^\infty x'(t)e^{-zt}dt + \int_0^\infty y'(t)e^{-zt}dt + 2\,X(z) + Y(z) = \frac{1}{z+2}.$$

Now

$$\int_0^\infty x'(t)e^{-zt}dt = x(t)e^{-zt}\Big|_0^\infty + z\,X(z)$$

$$= -\,x_0 + z\,X(z);$$

and

$$\int_0^\infty y(t)e^{-zt}dt = -\,y_0 + z\,Y(z).$$

Hence

$$(z-1)\,X(z) + (z+3)\,Y(z) = \frac{1}{z+1} + x_0 + y_0,$$

$$(z+2)\,X(z) + (z+1)\,Y(z) = \frac{1}{z+2} + x_0 + y_0.$$

One now solves for X(z) and Y(z) and then one uses the inverse transform to obtain x(t) and y(t).

11. The complementary function is

$$Ae^{zx/a} + Be^{-zx/a},$$

where, remember, A and B are functions of z. Denoting $\frac{d}{dx}$ by D we have

$$(D^2 - \frac{z^2}{a^2})U = - \frac{z}{a^2} g(x)$$

and so a particular integral is

$$- \frac{z}{a^2} \frac{1}{D^2-z^2/a^2} . g(x)$$

$$= - \frac{1}{2a} [\frac{1}{D-z/a} - \frac{1}{D+z/a}] g(x)$$

$$= - \frac{1}{2a} [e^{zx/a} \int_0^x e^{-zt/a} g(t)dt - e^{-zx/a} \int_0^x e^{zt/a} g(t)dt].$$

The general solution, therefore, is

$$U(x,z)=Ae^{zx/a}+Be^{-zx/a} - \frac{1}{2a}[e^{zx/a}\int_0^x e^{-zt/a}g(t)dt - e^{-zx/a}\int_0^x e^{zt/a}g(t)dt].$$

From $U(0, z) = 0$ we see that

$$A + B = 0.$$

Since

$$U(x, z) = [A - \frac{1}{2a} \int_0^x e^{-zt/a} g(t)dt] e^{zx/a} + [....]e^{-zx/a}$$

if $U(x, z) \to 0$ $(x \to \infty)$ for $\text{Re } z > c \geq 0$, then

$$A(z) = \frac{1}{2a} \int_0^\infty e^{-zt/a} g(t)dt.$$

[Note that there is a misprint and $x \to 0$ should be $x \to \infty$.]. Hence A, B are determined and the problem solved.

[Remark: This solution agrees with that given in Example 9.6.2 which is the same problem.]

12. The conditions are those of Example 9.6.2 with $u(x, 0) = g(x)$ replaced by $u(x, 0) = 0$ and $u_t(x, 0) = 0$ replaced by $u_t(x, 0) = h(x)$. In this case if

$$U(x, z) = \int_0^\infty u(x, t)e^{-zt}dt$$

then

$$U_{xx} - \frac{z^2}{a^2} U = - \frac{h(x)}{a^2}.$$

Hence the solution will be the same as on p.430 except that g(x) will be replaced by $\frac{h(x)}{z}$ and so, noting that we put $h(t) = -h(t)$ $(t > 0)$,

$$U(x,\ z) = \frac{1}{2}\cdot\frac{1}{z}\int_0^\infty h(x + a\eta)e^{-z\eta}\,d\eta + \frac{1}{2}\cdot\frac{1}{z}\int_0^\infty h(x - a\eta)e^{-z\eta}\,d\eta\,.$$

Now $L[1] = \frac{1}{z}$ and hence from the convolution result we obtain

$$u(x,\ t) = \frac{1}{2}\int_0^t h(x + a\tau)d\tau + \frac{1}{2}\int_0^t h(x - a\tau)d\tau\,.$$

13. Since the wave equation is linear if we can find solutions u_1, u_2, u_3 satisfying appropriate initial and boundary conditions so that

$$u = u_1 + u_2 + u_3$$

satisfies the conditions of the problem then this u will solve the problem. Consider then the following problems:

(i) $u_1(x,\ 0) = 0,\ \frac{\partial}{\partial t}u_1(x,\ t)\Big|_{t=0} = 0,\ u_1(0,\ t) = f(t)$ (Example 9.6.1);

(ii) $u_2(x,\ 0) = g(x),\ \frac{\partial}{\partial t}u_2(x,\ t)\Big|_{t=0} = 0,\ u_2(0,\ t) = 0$ (Example 9.6.2);

(iii) $u_3(x,\ 0) = 0,\ \frac{\partial}{\partial t}u_3(x,\ t)\Big|_{t=0} = h(x),\ u_3(0,\ t) = 0$ (Exercise 9.6.2).

14. We make the plausible assumption that if we consider a semi-infinite string with $x > 0$ and initial displacement $A \sin \pi x$, then the part between $x = 0$ and $x = 1$ will vibrate as though it were fixed at these two points. We therefore have the problem of Example 9.6.2 with $g(x) = A \sin \pi x$ and since $A \sin \pi x$ is an odd function

$$\begin{aligned} u(x,\ t) &= \frac{1}{2} A \sin [\pi(x + at)] + \frac{1}{2} \sin [\pi(x - at)] \\ &= A \sin \pi x . \cos a\pi t. \end{aligned}$$

This answer confirms that the above 'plausible' assumption is compatible with the mathematical model.

15. By the method discussed at the beginning of §9.7 if $U(z) = L[u(t)]$,

$$U(z) = L(1) + L[t^2]\ U(z).$$

Here we have used that the transform of the convolution of two functions is the product of the transforms (Theorem 9.2.7). Since

$$L(1) = \int_0^\infty e^{-zt}dt = \frac{1}{z},\ L(t^2) = \int_0^\infty t^2 e^{-zt}dt = \frac{2}{z^3} \quad (\text{Re } z > 0),$$

$$U(z) = \frac{1}{z} + \frac{2}{z^3}U(z)$$

$$U(z) = \frac{z^2}{z^3 - 2}.$$

U(z) comes from an 'unknown' function u(t) and assuming it has a Laplace transform this will be analytic for Re $z > c$ and some constant c. From the form of U(z) one sees that one can take $c = 2^{1/3}$. Note that a suitable value for c depends on the equation and not just on the half-planes of analyticity a rising from the particular functions that enter into it.

Hence

$$u(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{z^2}{z^3-2}\, e^{zt}dt,$$

where γ is a number satisfying $\gamma > 2^{1/3}$. Considering $\frac{1}{2\pi i}\int_{\gamma-iR}^{\gamma+iR}$ together with the integral over semi-circle on the left and letting $R \to \infty$ we find that

$$u(t) = \sum[\text{residues of } \frac{z^2}{z^3-2}.e^{zt}].$$

The residues arise from simple poles at $z = 2^{1/3}, 2^{1/3}\omega, 2^{1/3}\omega^2$, where $\omega = e^{2\pi i/3}$. Hence

$$u(t) = \frac{1}{3}[e^{2^{1/3}t} + e^{2^{1/3}\omega t} + e^{2^{1/3}\omega^2 t}]$$

$$= \sum_{k=0}^{\infty} \frac{2^k t^{3k}}{(3k)!}.$$

[Remark: It is a useful exercise to verify this result.]

16. In this problem u(t) is the 'unknown'. The right hand side of the equation is the convolution of u(t) and $\frac{1}{t^\alpha}$ and so

$$F(z) = L[f(t)] = L[u(t)].L[\frac{1}{t^\alpha}]$$

$$= U(z).\int_0^\infty \frac{e^{-zt}}{t^\alpha}dt.$$

Taking $z = x > 0$ and putting $\tau = xt$ we obtain

$$\int_0^\infty \frac{e^{-xt}}{t^\alpha}\,dt = x^{-(1-\alpha)}\int_0^\infty \tau^{(1-\alpha)-1}e^{-\tau}\,d\tau$$

$$= x^{-(1-\alpha)}.\Gamma(1-\alpha).$$

By continuation we note that

$$L\left(\frac{1}{t^{\alpha}}\right) = z^{-(1-\alpha)}.\Gamma(1-\alpha) \qquad (\mathrm{Re}\, z > 0),$$

where $z^{-(1-\alpha)}$ is taken to be real and positive on the positive real axis and is determined elsewhere in the right half plane by continuity. Hence

$$U(z) = \frac{z^{1-\alpha}F(z)}{\Gamma(1-\alpha)}.$$

One might consider that the right hand side arises from transforms having values $z^{1-\alpha}$ and $F(z)$. Now

$$L(t^{\beta}] = z^{-(1+\beta)}\Gamma(1+\beta)$$

and if one chooses β so that $-(1 + \beta) = 1 - \alpha$, i.e. $\beta = -(2 - \alpha)$, then the integral for $L[t^{\beta}]$ diverges at the lower limit. Hence, as in Example 9.7.2, we consider

$$U(z) = \frac{z}{\Gamma(1-\alpha)}.\frac{1}{z^{\alpha}}.F(z).$$

Now $L[\frac{1}{t^{1-\alpha}}] = \frac{\Gamma(\alpha)}{z^{\alpha}}$ (taking $\beta = -(1 - \alpha)$ above) and so, by the convolution theorem,

$$L^{-1}[\frac{\Gamma(\alpha)}{z^{\alpha}}.F(z)] = \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau ,$$

where L^{-1} denotes the inverse operator. Consequently, as in Example 9.7.2,

$$u(t) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau$$

$$= \frac{\sin \pi\alpha}{\pi} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau .$$

17. Let U_1, U_2 be the Laplace transforms of u_1, u_2; K_1, K_2, K_3, K_4 be the transforms of k_1, k_2, k_3, k_4; and F, G be the transforms of f, g. Then

$$U_1 = F + K_1U_1 + K_2U_2,$$

$$U_2 = G + K_3U_1 + K_4U_4.$$

and hence

$$(1 - K_1)U_1 + K_2U_2 = F,$$

$$- K_3U_1 + (1 - K_4)U_2 = G.$$

One now solves for U_1, U_2 and then applies the inverse transformation.

Chapter 10

Asymptotic Expansions

| READ §10.1 |

At the top of p.438 one really requires more than

$$a_{k+1} + \frac{h_k}{w_{k+1}} \neq 0.$$

What one requires is that for each k there is a neighbourhood of z_0 in which $a_{k+1} + \frac{h_k}{w_{k+1}}$ is bounded away from 0. From the conditions stated, in some neighbourhood of z_0,

$$\left|\frac{h_k}{w_{k+1}}\right| < \frac{|a_{k+1}|}{2}$$

and hence in this neighbourhood,

$$\left|a_{k+1} + \frac{h_k}{w_{k+1}}\right| \geq |a_{k+1}| - \left|\frac{h_k}{w_{k+1}}\right| > \frac{|a_{k+1}|}{2}.$$

Self-Assessment Questions

SAQ 1, 2, 3 are Exercise 10.1, #1, 3, 4.

| READ §10.2 |

Though the results of this section may not be unexpected the details of the proofs can be rather tricky.

Problems for §10.1, §10.2

P.1, 2, 3 are Exercises §10.1, #2, 5, 6.

P.4, is Exercise 10.2.

| READ §10.3 |

The details in this section are a bit overwhelming, but one should try to get hold of the ideas. I feel that being adept at the kind of calculations that arise is a bit like being highly trained for some sporting event. Most of us for most of the time are a bit flabby, but with a bit of common-sense and effort we can attain our goals if we take our time, and when it is necessary we can get into tip-top shape with a bit of training.

Theorem 10.3.1 In this theorem we have conditions on F(t) for t small and for t large. We consider for $\delta > 0$ small,

$$\int_0^\infty F(t)e^{-zt}dt = \int_0^\delta F(t)e^{-zt}dt + \int_\delta^\infty F(t)e^{-zt}dt$$

and use the two conditions on F(t) in $\int_0^\delta$ and in $\int_\delta^\infty$ respectively. In the proof of the theorem this splitting of $\int_0^\infty$ into $\int_0^\delta$ and $\int_\delta^\infty$ is not explicit, but it is coped with by the result for fixed N stated at the top of p.447. This incorporates the two conditions of the hypotheses. However, the splitting should be kept in mind as it may be useful in other contexts.

Example 10.3.1 It is perhaps a bit misleading to say near the bottom of p.447 that Watson's lemma is being applied to the integral above. This integral requires quite a bit of modification before Watson's lemma can be applied.

p.448 Near the bottom of the page one requires that σ, s are bounded away from 1, not just that σ, $s \neq 1$ for $u \geq \varepsilon > 0$. It is clear, in fact, that the stronger result is true for $u \geq \varepsilon > 0$.

pp.448, 449 The explanation given for the transformation relating ς, ω is somewhat obscure. One might be led to such a transformation by the following considerations. It is usually simpler to have a particular point at the origin and so we might see the effect of setting $s - 1 = \omega$. We are aware that we are looking for two solutions of $u = s - 1 - \log s$ and so we might put $2u = \varsigma^2$ and hope that these solutions will be coped with by the index 2; or rather we would note that for small ω,

$$\omega - \log(1 + \omega) = \frac{\omega^2}{2} - \frac{\omega^3}{3} + \dots$$

and then the appropriateness of $2u = \varsigma^2$ is clear.

Theorems 10.3.2, 10.3.3 One should aim at acquiring an understanding of these results in broad outline rather than spending a lot of time over the details. Note that one considers $\int_0^\infty = \int_0^\delta + \int_\delta^\infty$ in the proofs of these theorems.

p.452 At the top of the page the reference to Theorem 10.3.2 is perhaps inadequate. One has also to take account of the use of integration by parts as presented at the top of p.450.

p.453 The description of the steepest descent method is more or less correct, but the assertions made are not so obvious as the presentation might lead one to expect. The method requires one to look at the zeros of $h'(\varsigma)$ and show that through such a zero, ς_0 say, one can find an admissible path of integration on which $\mathrm{Re}[h(\varsigma)]$ decreases as quickly as possible from $\mathrm{Re}[h(\varsigma_0)]$. For large $x > 0$, $|e^{xh(\varsigma)}|$ will be relatively small on such a path except near to ς_0 and so one hopes the behaviour of the integral can be found from Watson's lemma. More generally one hopes this idea will work for Re z > 0 and appropriately restricted. The steepest descent method as presented here has modifications which extend its usefulness, but we shall not go into this topic.

Consider a point ς_0, where $h'(\varsigma)$ has a simple zero. We shall not consider other possibilities. Assume, without loss of generality, that $\varsigma_0 = 0$, $h(\varsigma_0) = 0$. If $\varsigma = \rho e^{i\phi}$, then near $\varsigma = 0$,

$$\begin{aligned} h(\varsigma) &= a_2\varsigma^2 + a_3\varsigma^3 + \ldots \qquad (a_1 \neq 0) \\ &= |a_2|\rho^2 e^{i(2\phi+\alpha)} + \ldots \qquad (\alpha = \arg a_2). \end{aligned}$$

If $a_3 = a_4 = \ldots = 0$, then

$$\mathrm{Re}[h(\varsigma)] = |a_2|\rho^2 \cos(2\phi + \alpha)$$

and the steepest descent will be when $2\phi + \alpha = \pm\pi$, i.e. $\phi = \frac{\pi - \alpha}{2}$, $\phi = -\frac{\pi + \alpha}{2}$. In general, when ρ is small the steepest descent path will leave $\varsigma_0 = 0$ in these directions.

Consider next a point ς on the path of steepest descent. We can take $\varsigma = 0$ and we suppose that $h'(0) \neq 0$. Then, if $\varsigma = \rho e^{i\phi}$, near $\varsigma = 0$,

$$h(\varsigma) = b_0 + b_1\varsigma + b_2\varsigma^2 + \dots \qquad (b_1 \neq 0)$$

$$= b_0 + |b_1|\,\rho e^{i(\phi+\beta)} + \dots \qquad (\beta = \arg b_1).$$

If $b_2 = b_3 = \dots = 0$, then

$$\text{Re}[h(\varsigma)] = \text{Re } b_0 + |b_1|\,\rho \cos(\phi+\beta)$$

and the path of steepest descent is when $\phi + \beta = \pi$, i.e. $\phi = \pi - \beta$. Now

$$\text{Im}[h(\varsigma)] = \text{Im } b_0 + |b_1|\,\rho \sin(\phi+\beta)$$

and

$$\frac{\partial}{\partial\rho}\,\text{Im}[h(\rho e^{i\phi})] = |b_1| \sin(\phi+\beta)$$

$$= 0 \qquad (\phi = \pi - \beta).$$

One sees in general that if the path of steepest descent leaves $\varsigma = 0$ in the direction ϕ_0, then

$$\frac{\partial}{\partial\rho}\,\text{Im}[h(\rho e^{i\phi})]\Big|_{\rho=0,\ \phi=\phi_0} = 0.$$

Hence, if $\varsigma = \varsigma(s)$ is the path of steepest descent parameterised by arc length, then

$$\frac{d}{ds}\,\text{Im}[h(\varsigma(s))] = 0,$$

and so $\text{Im}[h(\varsigma)]$ is constant on the path.

Example 10.3.3 The wording of this example is somewhat odd. It is strange that arg z is not considered till later. An example of such complexity as this seems rather inappropriate as an illustration of the steepest descent method.

p.457 One should consider the method of stationary phrase with caution. Its formal application may give the 'right' answer, but it may not be easy to justify the result. The conditions considered here are rather special.

Problems for §10.3

P5, 6, 7, 8, 9, 10, 11 are Exercises 10.3, #1, 2, 3, 4, 5, 6, 7.

| READ §10.4 |

This is a difficult section to get to grips with. You should note that the arguments are somewhat heuristic and justifications are adumbrated rather than dealt with properly.

pp.461, 462 It is not made clear what distinguishes normal solutions from subnormal solutions. It would seem that in the former $q_0 \neq 0$ and in the latter $q_0 = 0$ and that is all. Otherwise consider the forms of the solutions in the two cases?

p.462 You should note that γ_1, γ_2 are not arbitrary constants. They depend on the solution u. In fact

$$\gamma_1 = e^{2\alpha z} z^{-2\beta} \left.\frac{du}{dz}\right|_{z=z_1}, \quad \gamma_2 = u(z_1).$$

Also in the definition of $K(z, \eta)$, z_1 should be z.

p.463 The discussion preceding Example 10.4.1 is confused. It seems to indicate that one can treat the case $\gamma_1 = 0$ and supplement the result in this special case by adding on the term $-\gamma_1 \int_z^\infty e^{-2\alpha\zeta} \zeta^{2\beta} d\zeta$. However, it does not seem this can be done except, perhaps, in particular examples.

p.466 In the integral equation for $\eta(\xi)$ the initial conditions should be $c_2 = \eta(0)$, $c_1 = \eta'(0)/\lambda$.

It is not clear to me how the integral equation is obtained as a consequence of the results in Exercises 7.2. However, one can argue as follows in order to obtain the equation. Given the initial conditions there is a (unique) solution $\eta(\xi)$ of

$$\frac{d^2\eta}{d\xi^2} + \lambda^2\eta = r(\xi)\eta.$$

Let $\eta(\xi)$ denote this solution and consider the equation

$$\frac{d^2w}{d\xi^2} + \lambda^2 w = r(\xi)\eta,$$

i.e. we consider this equation for w with $r(\xi)\eta = r(\xi)\eta(\xi)$ as a given function of ξ. The complementary function, i.e. the solution of the homogeneous equation, is

$$w(\xi) = A \sin \lambda\xi + B \cos \lambda\xi \quad \text{(A, B constants)}.$$

If $D = \frac{d}{d\xi}$ then a particular integral comes from

$$w(\xi) = \frac{1}{D^2 + \lambda^2}[r(\xi)\eta(\xi)]$$

$$= \frac{1}{2i\lambda}\left(\frac{1}{D - i\lambda} - \frac{1}{D + i\lambda}\right)[r(\xi)\eta(\xi)]$$

$$= \frac{1}{2i\lambda}\int_0^\xi [e^{i\lambda(\xi - t)} - e^{-i\lambda(\xi - t)}]\, r(t)\eta(t)dt$$

$$= \frac{1}{\lambda}\int_0^\xi \sin[\lambda(\xi - t)]\, r(t)\eta(t)dt.$$

Now

$$w(0) = B,\ w'(0) = \lambda A$$

and if we impose the same initial conditions on w as on η then $w(\xi) - \eta(\xi)$ satisfies the homogeneous equation and $w(0) - \eta(0) = 0$, $w'(0) - \eta'(0) = 0$ and so $w(\xi) \equiv \eta(\xi)$ and hence η satisfies the integral equation.

Problems for §10.4

P12, 13, 14, 15 are Exercises 10.4, #1, 2, 3, 5.

Solutions to Self-Assessment Questions

1. It is enough to show that

$$f(z) = \sum_{n=1}^{N-1} a_n w_n(z) + o(w_{N-1}) \quad (z \to z_0),$$

since then one can work backwards down the sequence step by step. By hypotheses

$$f(z) = \sum_{n=1}^{N-1} a_n w_n(z) + a_N w_N(z) + o(w_N) \quad (z \to z_0)$$

and, since (w_n) is an asymptotic sequence,

$$w_N = o(w_{N-1}) \quad (z \to z_0).$$

Hence

$$f(z) = \sum_{n=1}^{N-1} w_n(z) + o(w_{N-1}) \quad (z \to z_0).$$

2. There is some number $R > 0$ such that

$$f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k \quad (|z - z_0| < R).$$

The series converges for $|z - z_0| = \frac{R}{2}$ and hence there is a constant K such that

$$|a_k| \left(\frac{R}{2}\right)^k \le K \quad (k = 0, 1, 2, \ldots).$$

Hence for $|z - z_0| = \frac{R}{4}$ and any positive integer N,

$$\left|\sum_{k=N+1}^{\infty} a_k(z - z_0)^k\right| \le |z - z_0|^{N+1} \sum_{k=N+1}^{\infty} K\left(\frac{2}{R}\right)^k . \left(\frac{R}{4}\right)^{k-N-1}$$

$$\le K.\left(\frac{2}{R}\right)^{N+1} |z - z_0|^{N+1} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$$

$$= 2K.\left(\frac{2}{R}\right)^{N+1} |z - z_0|^{N+1} .$$

Therefore, for $0 < |z - z_0| \le \frac{R}{4}$,

$$f(z) = \sum_{k=0}^{N} a_k(z - z_0)^k + g_N(z),$$

where

$$\frac{|g_N(z)|}{|z-z_0|^N} \le 2K.\left(\frac{2}{R}\right)^{N+1} |z - z_0|$$

$$\to 0 (z \to z_0).$$

Since the asymptotic sequence in this case is $w_n(z) = (z - z_0)^n$ this proves the desired result.

3. We have that

$$f(z) = \sum_{n=0}^{N} a_n w_n(z) + a_{N+1} w_{N+1}(z) + o(w_{N+1}) \quad (z \to z_0),$$

and since

$$a_{N+1} w_{N+1} + o(w_{N+1}) = o(w_{N+1})$$

the result follows.

Solutions to Problems

1. For each positive integer N,

$$1 - t^2 + t^4 \ldots . + (-1)^N t^{2N} = \frac{1 + (-1)^N t^{2N+2}}{1+t^2}$$

and so

$$\int_0^{\infty} \frac{e^{-zt}}{1+t^2} dt = \int_0^{\infty} e^{-zt}[1 - t^2 + \ldots + (-1)^N t^{2N}]dt + R_N(z),$$

where

$$R_N(z) = (-1)^{N+1} \int_0^\infty \frac{t^{2N+2}.e^{-zt}}{1+t^2}\,dt.$$

The integrals all converge since

$$\mathrm{Re} z \geq |z| \cos\left(\frac{\pi}{2} - \alpha\right) = |z| \sin \alpha$$

and

$$|e^{-zt}| = |e^{-\mathrm{Re} z.t}| \leq e^{-|z|\sin\alpha.t}.$$

Using integration by parts, for any positive integer k,

$$\int_0^\infty t^k e^{-zt}dt = -\frac{1}{z}t^k e^{-zt}\Big|_0^\infty + \frac{k}{z}\int_0^\infty t^{k-1}e^{-zt}dt$$

$$= \frac{k}{z} . \int_0^\infty t^{k-1}e^{-zt}dt.$$

Hence

$$\int_0^\infty t^k e^{-zt}dt = \frac{k!}{z^k} . \int_0^\infty e^{-zt}dt$$

$$= \frac{k!}{z^{k+1}}.$$

Therefore

$$\int_0^\infty e^{-zt}[1 - t^2 + \ldots + (-1)^N t^{2N}]\,dt$$

$$= \frac{1}{z} - \frac{2!}{z^3} + \frac{4!}{z^5} - \ldots + (-1)^N \frac{(2N)!}{z^{2N+1}}.$$

Also

$$|R_N(z)| \leq \left|\int_0^\infty \frac{t^{2N+2}}{1+t^2} e^{-zt}dt\right|$$

$$\leq \int_0^\infty t^{2N+2}\, e^{-|z|\sin\alpha.t}dt$$

$$= \frac{(2N+2)!}{(|z|\sin\alpha)^{2N+3}}.$$

Hence the result follows with

$$w_n(z) = \frac{1}{z^{2n+1}}.$$

2. First of all note that in R,

$$\mathrm{Re} z \geq |z| \sin \alpha > 0, \quad |z + t| \geq \mathrm{Re}(z + t) \geq |z| \sin \alpha \qquad (t \geq 0)$$

and so the integral is well defined. Integrating by parts

$$\int_0^z \frac{e^{-t}}{z+t}\,dt = -\frac{e^{-t}}{z+t}\Big|_0^\infty - \int_0^\infty \frac{e^{-t}}{(z+t)^2}\,dt$$

$$= \frac{1}{z} - [-\frac{e^{-t}}{(z+t)^2}\Big|_0^\infty - 2\int_0^\infty \frac{e^{-t}}{(z+t)^3}\,dt]$$

$$= \frac{1}{z} - \frac{1}{z^2} + 2\int_0^\infty \frac{e^{-t}}{(z+t)^3}\,dt$$

and so on. Hence for N a positive integer,

$$\int_0^z \frac{e^{-t}}{z+t}\,dt = \frac{1}{z} - \frac{1!}{z^2} + \frac{2!}{z^3} - \ldots + \frac{(-1)^{N-1}.N!}{z^{N+1}} + R_N(z),$$

where

$$R_N(z) = (-1)^N.N! \int_0^\infty \frac{e^{-t}}{(z+t)^{N+2}}\,dt.$$

Also

$$|R_N(z)| \le N! \int_0^\infty \frac{e^{-t}}{|z+t|^{N+2}}\,dt$$

$$\le \frac{N!}{(|z|\sin\alpha)^{N+2}} \int_0^\infty e^{-t}dt$$

$$= \frac{N!}{(\sin\alpha)^{N+2}}.\frac{1}{|z|^{N+2}}$$

and so the result follows.

3. Integrating by parts

$$\int_z^\infty \frac{e^{-t}}{t}\,dt = -\frac{e^{-t}}{t}\Big|_z^\infty - \int_z^\infty \frac{e^{-t}}{t^2}\,dt$$

$$= \frac{e^{-z}}{z} - \int_z^\infty \frac{e^{-t}}{t^2}\,dt,$$

where we consider the integral along a part of a ray from the origin in the complex t-plane. On the path of integration $|t| \ge |z| = r$ and $|e^{-t}| = e^{-|t|\cos\theta}$, $z = re^{i\theta}$, and so

$$\left|\int_z^\infty \frac{e^{-t}}{t^2}\,dt\right| \le \frac{1}{r^2}\int_r^\infty e^{-\rho\cos\theta}\,d\rho$$

$$= \frac{1}{\cos\theta}.\frac{e^{-\rho\cos\theta}}{r^2}$$

$$\le \frac{1}{\sin\alpha}.\left|\frac{e^{-z}}{z}\right|.\frac{1}{|z|}$$

and hence the result follows.

4. We interpret

$$f(z) \sim g(z) \ (z \in R, \ z \to z_0)$$

as

$$\frac{f(z)}{g(z)} \to 1 \ (z \in R. \ z \to z_0).$$

In this case

$$\left[\frac{f(z)}{g(z)}\right]^n \to 1 \ (z \in R, \ z \to z_0)$$

and so $[f(z)]^n \sim [g(z)]^n$.

Let $R = \mathbb{C}$, the complex plane, $z_0 = \infty$ and $f(z) = z^2 + z$, $g(z) = z^2$. Then $f(z) \sim g(z)$, but

$$\frac{e^{f(z)}}{e^{g(z)}} = e^{f(z)-g(z)} = e^z \not\to 1 (z \to \infty)$$

and so $e^{f(z)}$ is not asymptotic to $e^{g(z)}$.

Let $R = \{z\colon |z| > 1\}$, $z_0 = \infty$ and $f(z) = 1 + \frac{1}{z}$, $g(z) = 1 + \frac{1}{z^2}$. Then $f(z) \sim g(z)$, but

$$\frac{\log f(z)}{\log g(z)} = \frac{\log (1 + 1/z)}{\log (1 + 1/z^2)} = \frac{1/z - 1/2z^2 + \ldots}{1/z^2 - 1/2z^4 + ..} \ (z \to \infty)$$
$$= z[1 + o(1)] \ (z \to \infty)$$
$$\not\to 1 \ (z \to \infty)$$

and so $\log f(z)$ is not asymptotic to $\log g(z)$.

5. This is rather a tricky problem whose solution we shall only outline. Replacing u by t on p.448 et seq we consider $F(t) = \frac{d\sigma}{dt} - \frac{ds}{dt}$. Then $F(\zeta)$ is analytic in $\{\mathrm{Re}\ \zeta > 0\}$ and behaves 'nicely' in $\{|\arg \zeta| \le \frac{\pi}{2} - \delta\}$ for any δ with $0 < \delta < \frac{\pi}{2}$. Consider $z = x > 0$ and $\int e^{-z\zeta} F(\zeta)\, d\zeta$ round the contour consisting of the segment of the real ζ-axis from 0 to R, the arc of $\{|\zeta| = R\}$ from $\zeta = R$ to $\zeta = Re^{i\beta}$ with $\beta = -(\frac{\pi}{2} - \frac{\alpha}{2})$ and the segment of the ray $\arg \zeta = \beta$ from $Re^{i\beta}$ to the origin. Letting $R \to \infty$ the integral over the arc tends to 0 and so

$$\int_0^\infty e^{-xte^{i\beta}} F(te^{i\beta})e^{i\beta}dt = \int_0^\infty e^{-xt} F(t)\, dt.$$

Replacing x by z with $0 \le \arg z \le \pi - \alpha$ and noting that

$$-(\frac{\pi}{2} - \frac{\alpha}{2}) \le \arg [ze^{i\beta}] \le \pi - \alpha - (\frac{\pi}{2} - \frac{\alpha}{2}) = \frac{\pi}{2} - \frac{\alpha}{2}$$

we see that when multiplied by $z^z e^{-z}$ the integral on the left with x replaced by z continues the Γ-function into the sector $\{0 \le \arg z \le \pi - \alpha\}$.

For small t we find that $F(te^{i\beta})$ has the same kind of expansion as F(t) except that t is now replaced by $te^{i\beta}$. If we now use Theorem 10.3.1 with z replaced by $ze^{i\beta}$ then we obtain the desired result. The sector $\{-(\pi-\alpha) \le \arg z \le 0\}$ is dealt with in a similar fashion.

6. Since $n! = \Gamma(n + 1) = n\,\Gamma(n)$ from the asymptotic formula for $\Gamma(z)$ on p.449 we obtain

$$n! \sim n.\sqrt{2\pi}.e^{-n}n^{n-1/2}$$
$$= \sqrt{2\pi n}.n^n.e^{-n}.$$

7. This is the integral given on p.447 since

$$se^{1-s} = e^{1-s+\log s}$$

if we make the correction to the sign of log t. We proceed in a simple way as in the proof of Example 10.3.1. Putting t − 1 = u we find that

$$\int_0^\infty e^{z(1-t+\log t)}dt = \int_{-1}^\infty e^{z[-u+\log(1+u)]}du.$$

We now use the monotonicity of $-u + \log(1 + u)$ noted in Example 10.3.1 and

$$-u + \log(1 + u) = - u + u - \frac{u^2}{2} + \dots \qquad (|u| < 1).$$

Hence we obtain as $z \to \infty$ with $|\arg z| \le \frac{\pi}{2} - \alpha$,

$$\int_{-1}^\infty e^{z[-u+\log(1+u)]}du \sim 2\int_0^\infty e^{-zu^2/2}du$$
$$= 2.\sqrt{\frac{2}{z}}\int_0^\infty e^{-t^2}dt$$
$$= \sqrt{\frac{2\pi}{z}}.$$

To evaluate the integral we can take z = x > 0 at first and then use analytic continuation. If $t^2 = s$, then

$$\int_0^\infty e^{-t^2}\,dt = \frac{1}{2}\int_0^\infty s^{-1/2}e^{-s}\,ds = \frac{1}{2}\,\Gamma(\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2}.$$

8. Assume $|\arg z| \le \frac{\pi}{2} - \alpha$. One shows as in P5 that

$$\int_0^\infty t^z e^{-zt}\,dt = \int_0^{\infty e^{-i\theta}} \zeta^z\, e^{-z\zeta}\,d\zeta \qquad (\theta = \arg z),$$

where the second integral is along the ray $\arg\zeta = -\theta$ and $\zeta^z = e^{z\log\zeta}$ with $\log\zeta$ analytic in the right half ζ-plane and $\log\zeta$ is real for ζ real and positive. Hence

$$\int_0^\infty t^z e^{-zt}\,dt = \int_0^\infty (\rho e^{-i\theta})^z e^{-|z|\rho}.e^{-i\theta}\,d\rho$$

$$= e^{-i(z+1)\theta}\int_0^\infty \rho^z e^{-|z|\rho}\,d\rho.$$

Put $|z|\rho = u$ and then

$$\int_0^\infty t^z e^{-zt}\,dt = e^{-i(z+1)\theta}\,.\int_0^\infty \left(\frac{u}{|z|}\right)^z.e^{-u}\,\frac{du}{|z|}$$

$$= \frac{1}{z^{z+1}}\int_0^\infty u^z.e^{-u}\,du$$

$$= \frac{\Gamma(z+1)}{z^{z+1}}.$$

$$= \frac{\Gamma(z)}{z^z}.$$

Hence the asymptotic expansion of the integral follows from that of $\Gamma(z)$.

9. Consider first of all

$$\int_0^\infty e^{it^2}\,dt = \frac{1}{2}\int_0^\infty \frac{e^{iu}}{\sqrt{u}}\,du.$$

The latter integral clearly converges at the lower limit and integrating $\int_1^\infty \frac{e^{iu}}{\sqrt{u}}\,du$ by parts shows that it also converges at the upper limit.

Consider next

$$\int e^{i\zeta^2}\,d\zeta$$

round the contour consisting of the segment $[0, R]$ of the real axis, the arc $\{Re^{i\phi} : 0 \le \phi \le \frac{\pi}{4}\}$ and the segment $[Re^{i\pi/4}, 0]$ of the ray $\{\arg\zeta = \frac{\pi}{4}\}$. The integral over the arc is

$$\int_0^{\pi/4} e^{iR^2(\cos2\phi+i\sin2\phi)}.iRe^{i\phi}d\phi.$$

Now $\sin 2\phi \geq \frac{2}{\pi}.2\phi = \frac{4\phi}{\pi}$ for $0 \leq \phi \leq \frac{\pi}{4}$ and so the modulus of the integral above is bounded by

$$R\int_0^{\pi/4} e^{-R^2\sin2\phi}d\phi \leq R\int_0^{\pi/4} e^{-4R^2\phi/\pi}d\phi$$

$$= \frac{\pi}{4R}(1 - e^{-R^2})$$

$$\to 0 \quad (R \to \infty)$$

and so

$$\int_0^\infty e^{it^2}dt = e^{i\pi/4}\int_0^\infty e^{-u^2}du$$

$$= e^{i\pi/4}.\frac{\sqrt{\pi}}{2}.$$

Therefore

$$C(x) + i\,S(x) = \int_0^\infty e^{it^2}dt - \int_x^\infty e^{it^2}dt$$

$$= \frac{\sqrt{\pi}}{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) - \int_x^\infty e^{it^2}dt.$$

In $\int_x^\infty e^{it^2}dt$ put $t^2 = u$ and then

$$\int_x^\infty e^{it^2}dt = \frac{1}{2}\int_{x^2}^\infty \frac{e^{iu}}{u^{1/2}}du.$$

Integrating by parts we obtain

$$\int_{x^2}^\infty \frac{e^{iu}}{u^{1/2}}du = \frac{1}{i}\frac{e^{iu}}{u^{1/2}}\Bigg|_{x^2}^\infty + \frac{1}{2i}\int_{x^2}^\infty \frac{e^{iu}}{u^{3/2}}du$$

$$= i\,\frac{e^{ix^2}}{x} + \frac{1}{2i}\left[\frac{e^{iu}}{i.u^{3/2}}\Bigg|_{x^2}^\infty + \frac{3}{2i}\int_{x^2}^\infty \frac{e^{iu}}{u^{5/2}}du\right]$$

and so on. After the above steps

$$\int_{x^2}^\infty \frac{e^{iu}}{u^{1/2}}du = i\,\frac{e^{ix^2}}{x} + \frac{1}{2}\frac{e^{ix^2}}{x^3} - \frac{3}{2^2}\int_{x^2}^\infty \frac{e^{iu}}{u^{5/2}}du.$$

Hence

$$C(x) = \frac{1}{2}\sqrt{\frac{\pi}{2}} - \mathrm{Re}[\frac{1}{2}\int_{x^2}^{\infty} \frac{e^{iu}}{u^{1/2}}\,du]$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{2}} - \frac{1}{2}[-\frac{\sin x^2}{x} + \frac{1}{2}\frac{\cos x^2}{x^3} - \frac{3}{2^2}\int_{x^2}^{\infty}\frac{\cos u}{u^{5/2}}\,du],$$

with a corresponding result for S(x).

For any positive integer n,

$$|\int_{x^2}^{\infty}\frac{e^{iu}}{u^{n+1/2}}\,du| \le \int_{x^2}^{\infty}\frac{du}{u^{n+1/2}}$$

$$= \frac{2}{2n-1}\cdot\frac{1}{x^{2n-1}}.$$

Consequently the result of repeated integration by parts together with the inequality above shows that the series of the problem are asymptotic series.

10. Note from p.316 that

$$J_n(iy) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta - y\sin\theta}\,d\theta .$$

When y is large and positive, then one expects that the integral will be dominated by the contribution over a small interval about $-\frac{\pi}{2}$. Put

$\theta = -\frac{\pi}{2} + \phi$ and consider

$$J_n(iy) = \frac{1}{2\pi}\int_{-\pi/2}^{3\pi/2} e^{-in(-\pi/2+\phi)+y\cos\phi}\,d\phi$$

$$= \frac{i^n}{2\pi}\int_{-\pi/2}^{3\pi/2} e^{-in\phi+y\cos\phi}\,d\phi .$$

Since $\cos\phi = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} \ldots$, given $\varepsilon > 0$ for all small $\delta > 0$,

$$1 - (1+\varepsilon)\frac{\phi^2}{2} \le \cos\phi \le 1 - (1-\varepsilon)\frac{\phi^2}{2} \qquad (|\phi| \le \delta).$$

For $-\frac{\pi}{2} \le \phi \le -\delta$ and for $\delta \le \phi \le \frac{3\pi}{2}$, assuming δ is small enough,

$$\cos\phi = 1 - 2\sin^2\frac{\phi}{2} \le 1 - 2\sin^2\frac{\delta}{2} \le 1 - 2\left(\frac{\delta}{\pi}\right)^2 .$$

Also, for $|\phi| \le \delta$,

$$|e^{-in\phi} - 1| = 2\,|\sin\frac{n\phi}{2}| \le n\delta .$$

The proof of the asymptotic formula requires a number of steps.

(i) $$\left|\left(\int_{-\pi/2}^{-\delta} + \int_{\delta}^{3\pi/2}\right) e^{-in\phi+y\cos\phi} d\phi \right| \le 2\pi . e^{y[1-2(\delta/\pi)^2]}$$

$$= 2\pi e^{-2y(\delta/\pi)^2} . e^{y}.$$

(ii) $$\int_{-\delta}^{\delta} e^{-in\phi+y\cos\phi} d\phi = \int_{-\delta}^{\delta} e^{y\cos\phi} d\phi + \int_{-\delta}^{\delta} (e^{-in\phi} - 1).e^{y\cos\phi} d\phi$$

and

$$\left|\int_{-\delta}^{\delta} (e^{-in\phi} - 1)e^{y\cos\phi} d\phi\right| \le n\delta \int_{-\delta}^{\delta} e^{y\cos\phi} d\phi .$$

(iii) Consider next

$$e^{y} \int_{-\delta}^{\delta} e^{-y(1+\varepsilon)\phi^2/2} d\phi \le \int_{-\delta}^{\delta} e^{y\cos\phi} d\phi \le e^{y} \int_{-\delta}^{\delta} e^{-y(1-\varepsilon)\phi^2/2} d\phi .$$

(iv) $$\int_{-\delta}^{\delta} e^{-y(1+\varepsilon)\phi^2/2} d\phi = \int_{-\infty}^{\infty} e^{-y(1+\varepsilon)\phi^2/2} d\phi + A,$$

where

$$A = 2 \int_{\delta}^{\infty} e^{-y(1+\varepsilon)\phi^2/2} d\phi$$

$$\le 2 \int_{\delta}^{\infty} e^{-y(1+\varepsilon)\delta . \phi/2} d\phi$$

$$= \frac{4}{y(1+\varepsilon).\delta} . e^{-y(1+\varepsilon)\delta^2/2} .$$

(v) $$\int_{-\infty}^{\infty} e^{-y(1+\varepsilon)\phi^2/2} d\phi = 2 \int_{0}^{\infty} e^{-y(1+\varepsilon)\phi^2/2} d\phi$$

$$= \sqrt{\frac{2\pi}{y(1+\varepsilon)}},$$

as in the solution to P7.

I will not put everything down in detail, but indicate from the above how one arrives at the desired answer. Considering the integral over $[-\frac{\pi}{2}, \frac{3\pi}{2}]$ for $J_n(iy)$ one sees from (i) that for $y > 0$ large we can restrict attention to $[-\delta, \delta]$ for any small $\delta > 0$. (ii) shows that if δ is small then the effect of replacing $e^{-in\phi}$ in the integral over $[-\delta, \delta]$ by 1 is 'negligible'. (iii), (iv) and (v) shows that when δ is small and y is large the effect of replacing $\int_{-\delta}^{\delta} e^{y\cos\phi}d\phi$ by $e^y \int_{-\infty}^{\infty} e^{-y\phi^2/2}d\phi$ is 'negligible'. The consequence of the above is that as $y \to \infty$,

$$J_n(iy) \sim \frac{i^n e^y}{2\pi} \int_{-\infty}^{\infty} e^{-y\phi^2/2}d\phi$$

$$= \frac{i^n e^y}{\sqrt{2\pi y}} .$$

11. We assume that in $\int_z^{\infty} e^{-t}t^{-\nu}dt$ the integration is along the ray in the complex t-plane given by arg t = arg z. Integrating by parts one obtains

$$\int_z^{\infty} e^{-t}t^{-\nu}dt = - e^{-t}t^{-\nu} \Big|_z^{\infty} - \nu \int_z^{\infty} e^{-t}.t^{-\nu-1}dt$$

$$= e^{-z}.z^{-\nu} - [-\nu e^{-t}.t^{-\nu-1} \Big|_z^{\infty} - \nu(\nu + 1) \int_z^{\infty} e^{-t}.t^{-\nu-2}dt]$$

$$= e^{-z} [z^{-\nu} + \nu z^{-\nu-1}] + \nu(\nu + 1) \int_z^{\infty} e^{-t}.t^{-\nu-2}dt.$$

[Note that $a_0 = 1$ and not 0 as stated in Dettman].

After an appropriate number of integrations by parts we have to deal with a term involving

$$\int_z^{\infty} e^{-t}t^{-\nu-n}dt$$

and

$$|\int_z^{\infty} e^{-t}t^{-\nu-n}dt| \le |e^{-z}| \int_{|z|}^{\infty} t^{-\nu-n}dt$$

$$= \frac{|z|^{-\nu-n+1}.|e^{-z}|}{\nu+n-1}.$$

Therefore the result follows.

12. From $w = ue^{-1/2\int p}$ we obtain

$$w' = (u' - \tfrac{1}{2}pu)e^{-1/2\int p},$$

$$w'' = [(u'' - \tfrac{1}{2}p'u - \tfrac{1}{2}pu') - \tfrac{1}{2} p(u' - \tfrac{1}{2}pu)]e^{-1/2\int p}.$$

Using these results for w, w', w" in $w'' + pw' + qw = 0$ gives the equation for u.

From the solutions to P3 and P7 in Chapter 7 we see that the relationships given for behaviour at ∞ do in fact hold.

13. From $z = \zeta^2$ and $w(z) = \frac{u(\zeta)}{\zeta^2}$ we obtain

$$w'(z) = [\frac{u'(\zeta)}{\zeta^2} - \frac{2u(\zeta)}{\zeta^3}].(\frac{dz}{d\zeta})^{-1}$$

$$= \frac{u'(\zeta)}{2\zeta^3} - \frac{u(\zeta)}{\zeta^4},$$

$$w''(z) = [\frac{u''(\zeta)}{2\zeta^3} - \frac{3}{2}.\frac{u'(\zeta)}{\zeta^4} - \frac{u'(\zeta)}{\zeta^4} + \frac{4u(\zeta)}{\zeta^5}]\ (\frac{dz}{d\zeta})^{-1}$$

$$= \frac{u''(\zeta)}{4\zeta^4} - \frac{5}{4}.\frac{u'(\zeta)}{\zeta^5} + \frac{2u(\zeta)}{\zeta^6}.$$

Hence

$$u''(\zeta) - 5.\frac{u'(\zeta)}{\zeta} + [\frac{8}{\zeta^2} + 4\zeta^2 Q(\zeta)]\ u(\zeta) = 0,$$

where

$$Q(\zeta) = q(\zeta^2) = \frac{q_1}{\zeta^2} + \frac{q_2}{\zeta^4} + \ldots \qquad (q_1 \neq 0).$$

It follows that

$$\frac{8}{\zeta^2} + 4\zeta^2 Q(\zeta) = 4q_1 + \frac{8+4q_2}{\zeta^2} + \ldots\ .$$

Applying the argument on p.461 to the equation for $u(\zeta)$ we see that in the present case $\alpha \neq 0$, $\beta = 0$. This follows from the relations near the bottom of p.461 with q_0, q_1 there replaced by $4q_1 \neq 0$ and 0. Hence we get a 'solution'

$$u(\zeta) = e^{\alpha\zeta} \sum_{k=0}^{\infty} c_k\ \zeta^{-k}$$

and in terms of z and w(z) we find that

$$\sqrt{z}\ w(z) = e^{\alpha\sqrt{z}} \sum_{k=0}^{\infty} c_k\ \zeta^{-k}$$

and so

$$w(z) = e^{\alpha\sqrt{z}} \sum_{k=0}^{\infty} c_k\ z^{-1/2-k/2}.$$

[The answer given by Dettman seems to be wrong.]

14. The argument required to establish this result is very similar to that used on pp.343, 344 to deal with the Fredholm integral equation. (Note,

incidentally, that in the formula in the middle of p.465 that verifies that v(z) is a solution the 2 in the first expression should be η^2 and the 1 in the second expression should be omitted.)

Let

$$H[f] = H[f](z) = \int_z^\infty K(z, \eta).\frac{\nu^2-1/4}{\eta^2}.f(\eta)d\eta .$$

In this notation, with v_k as specified on p.464,

$$f_1 = 1 + H[f],$$

$$f_2 = 1 + H[f_1] = 1 + H[1 + H[f]]$$

$$= 1 + H[1] + H^2[f]$$

$$= v_1 + H^2[f],$$

$$f_3 = 1 + H[v_1 + H^2[f]$$

$$= v_2 + H^3[f],$$

and so on. Assuming f is bounded it follows as on p.464 that

$$|H[f]| \le \frac{M}{|z|}, \quad |H^2[f]| \le \frac{M^2}{2!}.\frac{1}{|z|^2}, \ldots$$

and so

$$|H^n[f]| \le \frac{M^n}{n!}.\frac{1}{|z|^n} \to 0 \quad (n \to \infty)$$

for any fixed $z \neq 0$. Therefore, for any such z,

$$f_{n+1}(z) - v_n(z) = H^{n+1}[f](z) \to 0 \qquad (n \to \infty).$$

Consequently the sequences (f_n) and (v_n) converge to the same function.

Uniqueness follows by considering two possible solutions v(z), V(z). If we start the iteration process with f(z) = V(z) then by the above $f_n \to v$ $(n \to \infty)$. But in this case $f_n = V$ for all n and so V = v.

15. We have that

$$J_\nu(x) = \sum_{n=0}^\infty \frac{(-1)^n(x/2)^{2n+\nu}}{n!\Gamma(\nu+n+1)}$$

$$= \frac{1}{\Gamma(\nu+1)}.(\frac{x}{2})^n \; [1 + \sum_{n=1}^\infty \frac{(-1)^n(x/2)^{2n}}{n!(\nu+1)(\nu+2)\ldots(\nu+n)}]$$

and

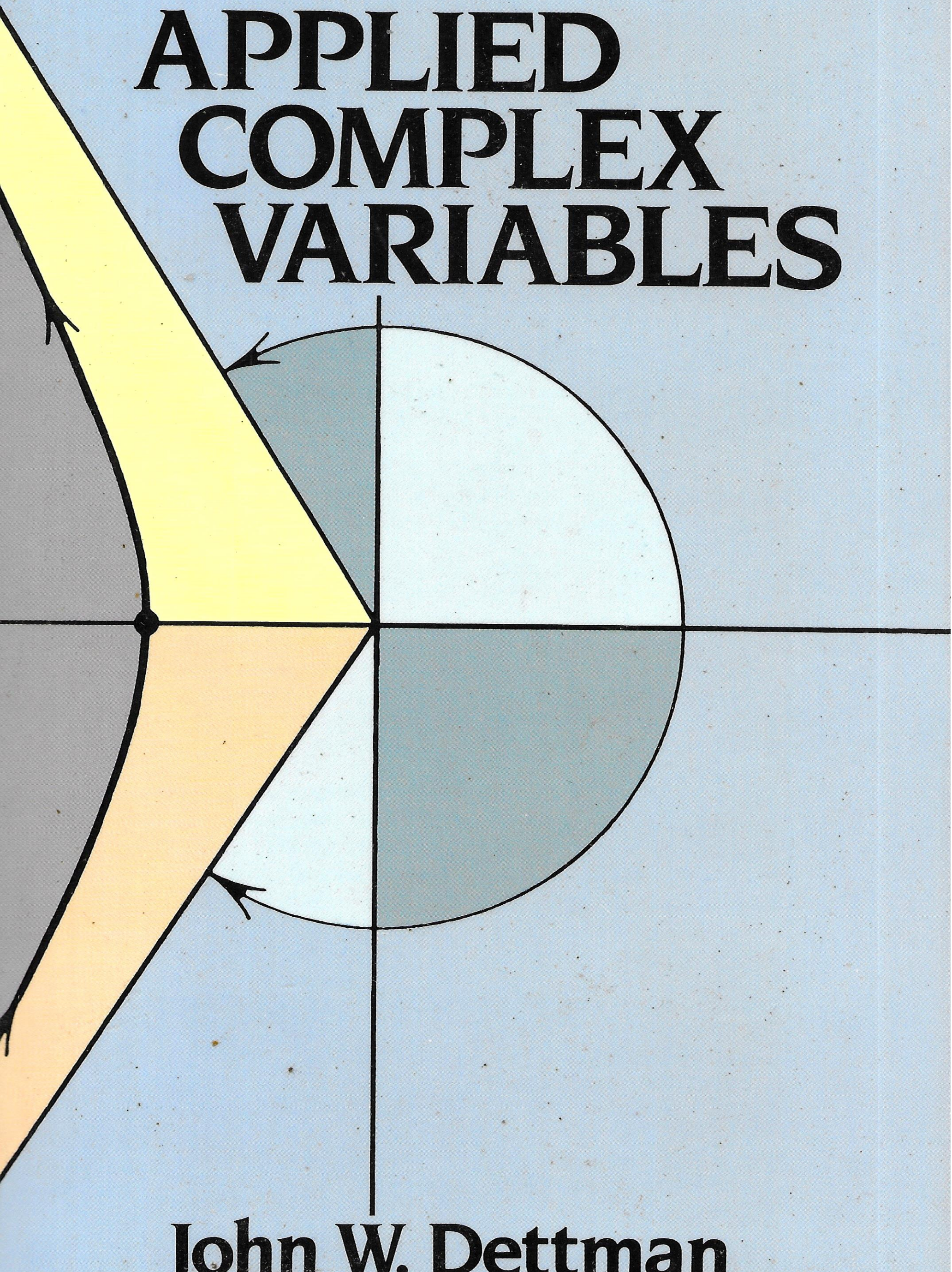
$$\left|\sum_{n=1}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n!(\nu+1)\ldots(\nu+n)}\right| \le \frac{1}{\nu+1} \sum_{n=1}^{\infty} \frac{(|x|/2)^n}{n!}$$

$$= \frac{1}{\nu+1}\,[e^{(|x|/2)^2} - 1].$$

Hence the result follows.

Printed in Great Britain by
The Open University

APPLIED COMPLEX VARIABLES



John W. Dettman

APPLIED COMPLEX VARIABLES

John W. Dettman

Analytic function theory is a traditional subject going back to Cauchy and Riemann in the 19th century. Once the exclusive province of advanced mathematics students, its applications have proven vital to today's physicists and engineers. In this highly regarded work, Professor John W. Dettman offers a clear, well-organized overview of the subject and various applications—making the often-perplexing study of analytic functions of complex variables more accessible to a wider audience.

The first half of *Applied Complex Variables*, designed for sequential study, is a step-by-step treatment of fundamentals, presenting superior coverage of concepts of complex analysis, including: the complex number plane; functions and limits; the Cauchy-Riemann conditions for differentiability; Riemann surfaces; the definite integral; power series; meromorphic functions; and much more. The second half provides lucid exposition of five important applications of analytic function theory, each approachable independently of the others: potential theory; ordinary differential equations; Fourier transforms; Laplace transforms; and asymptotic expansions. Helpful exercises are included at the end of each topic in every chapter.

The two-part structure of *Applied Complex Variables* affords the college instructor maximum classroom flexibility. Once fundamentals are mastered, applications can be studied in any sequence desired. Depending on how many are selected for study, Professor Dettman's impressive text is ideal for either a one- or a two-semester course. And, of course, the ambitious student possessing a knowledge of basic calculus will find its straightforward approach rewarding to his independent study efforts.

Applied Complex Variables is a cogent, well-written introduction to an important and exciting branch of advanced mathematics—serving both the theoretical needs of the mathematics specialist and the applied math needs of the physicist and engineer. Students and teachers alike will welcome this timely, moderately priced reissue of a widely respected work.

Unabridged and unaltered Dover republication (1984) of the work originally published by The Macmillan Company, New York, 1965. Preface. Contents. Bibliography. Index. 66 figures. Exercises at end of each topic. 512pp. 5⅜ x 8½. Paperbound.

Cover design by Paul E. Kennedy

See every Dover book in print at
www.doverpublications.com

ISBN 0-486-64670-X

90000

9 780486 646701

$18.95 IN USA
$31.95 IN CANADA